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CONTENTS

AVANT-PROPOS

This handout is a detailed course covering the official program of the Distributions Theory, primarily intended for students in the first year of the Master's program in Mathematical Analysis and Applications within the framework of the L.M.D. system. However, it may also be beneficial for students in the first year of the Master's programs in Partial Differential Equations, Stochastic Modeling and Forecasting in Operations Research, and Dynamic Systems and Geometry.

The recommended prerequisite knowledge includes: measure and integration theory, the concept of derivative in the classical sense, Taylor's formulas, sequences and series of functions.

INTRODUCTION

The main goal of this course is to introduce the student to the theory of distributions.

At the end of each Chapter, you will find a selection of typical exercises with detailed solutions, written progressively and in detail to help the student become familiar with new concepts and ensure the proper assimilation of key points.

The function spaces, which are almost essential ingredients for presenting the theory of distributions, are introduced in the first Chapter.

In Chapter two, we introduce the concept of distributions. We define the derivative of a distribution and demonstrate that any distribution can be differentiated an arbitrary number of times. We provide several examples of distributions and their derivatives, both on the real line and in higher dimensions.

In Chapter three, we introduce sequences and series of distributions. The concept of convergence in these spaces and other properties are also presented.

In Chapter 4, we delve into operations on distributions. Several operations defined for functions can be extended in the same way to distributions. We also define the concept of multiplying a distribution by a C^∞ function, which is a crucial ingredient when considering differential equations with variable coefficients. The notion of the support of a distribution is also presented.

CHAPTER 1

TEST FUNCTIONS AND CONVERGENCE IN $\mathcal{D}(\Omega)$

1.1 Introduction

In this chapter, we briefly review Lebesgue spaces and their properties. We introduce the spaces of test functions, smoothing sequences, exhaustive sequences, and the partition of unity theorem, which will be useful in the sequel.

Chapter Objectives

- Recall some functional spaces and preliminary results
- Understand the various properties of test functions
- Know how to study convergence in the space of test functions

Notation

Let us establish some notations that we will use throughout this course.

- Let $x = (x_1, \cdots, x_N)$ a point from $\mathbb{R}^N$ and $\|x\|$ its Euclidean norm.
- We denote **multi-indice** the element $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$.
- $|\alpha| = \alpha_1 + \cdots + \alpha_N$ is called the **length** α.
- We set $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for all $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$.
- For all multi-indices α and β, then

$$\alpha \leqslant \beta \Leftrightarrow \alpha_i \leqslant \beta_i, \ \forall i \in \{1, \cdots, N\}.$$

- Let $\alpha! = \alpha_1! \cdots \alpha_N!$ for $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$
- $(x+y)^\alpha = \sum_{\beta \leqslant \alpha} C_\alpha^\beta x^{\alpha-\beta} y^\beta = \sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} x^{\alpha-\beta} y^\beta, \quad \forall x, y \in \mathbb{R}^N$
- Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a function, for all $\alpha \in \mathbb{N}^N$

$$D^\alpha f = \partial^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} f = \frac{\partial^{|\alpha| f}}{\partial x_1{}^{\alpha_1} \cdots \partial x_N{}^{\alpha_N}}.$$

- Leibniz's formula:

$$D^\alpha (f \cdot g) = \sum_{\beta \leqslant \alpha} C_\alpha^\beta D^{\alpha-\beta} f D^\beta g.$$

1.2 Recalls on Lebesgue Spaces L^p

Let Ω be an open set in $\mathbb{R}^N$.

1.1 Definition

$L^1(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |f(x)|\, dx < \infty.$$

1.1 Theorem (Tonelli's Theorem)

Let Ω' be an open set in $\mathbb{R}^N$, and let $f : (x, y) \in \Omega \times \Omega' \mapsto F(x, y)$ be a measurable function. We assume that:

1) $\int_{\Omega'} |f(x, y)|\, dy < \infty$ *almost everywhere in* Ω.

2) $\int_{\Omega} \left(\int_{\Omega'} |f(x, y)|\, dy \right) dx < \infty$.

Then, $f \in L^1(\Omega \times \Omega')$.

Proof. See [12]. □

1.2 Theorem (Fubini's Theorem)

Let Ω' be an open set of $\mathbb{R}^N$, and let $f : (x, y) \in \Omega \times \Omega' \mapsto F(x, y)$ be a measurable function. If $f \in L^1(\Omega \times \Omega')$, then

1) $\int_{\Omega'} |f(x, y)|\, \mathrm{d}y < \infty$ *almost everywhere in* Ω,

2) $\int_{\Omega} \left(\int_{\Omega'} |f(x, y)|\, \mathrm{d}y \right) \mathrm{d}x < \infty$,

and

3) $\int_{\Omega} |f(x, y)|\, \mathrm{d}x < \infty$ *almost everywhere in* Ω',

4) $\int_{\Omega'} \left(\int_{\Omega} |f(x, y)|\, \mathrm{d}x \right) \mathrm{d}y < \infty$.

Moreover, we have

$$\int_{\Omega}\left(\int_{\Omega'}|f(x,y)|\,\mathrm{d}y\right)\mathrm{d}x=\int_{\Omega'}\left(\int_{\Omega}|f(x,y)|\,\mathrm{d}x\right)\mathrm{d}y. \tag{1.1}$$

Proof. See [12]. □

Remark 1.1. *In the particular case where f is positive as soon as one of the integrals in (1.1) is finite, then $f\in L^1(\Omega\times\Omega')$. If f has any sign (and f is not in $L^1(\Omega\times\Omega')$), the integrals in (1.1) can be finite without being equal.*

Example 1.1.

$$\int_0^1\left(\int_0^1\frac{x^2-y^2}{x^2+y^2}\,\mathrm{d}y\right)\mathrm{d}x=\frac{\pi}{4}\quad \textit{and}\quad \int_0^1\left(\int_0^1\frac{x^2-y^2}{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y=-\frac{\pi}{4}.$$

1.2 Definition (Lebesgue space L^p)

1) *Let $1\leqslant p<\infty$, $L^p(\Omega)$ is the space of measurable functions $f:\Omega\to\mathbb{R}$ such that $\int_{\Omega}|f(x)|^p\,\mathrm{d}x<\infty$*

2) *$L^\infty(\Omega)$ is the space of measurable functions $f:\Omega\to\mathbb{R}$ essentially bounded,i.e., exists a constant $C>0$ such that $|f(x)|\leqslant C$, almost everywhere $x\in\Omega$.*

1.3 Theorem

Let $1\leqslant p\leqslant\infty$, $L^p(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(\Omega)}=\left(\int_{\Omega}|f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}},\quad 1\leqslant p<\infty,$$

where

$$\|f\|_{L^\infty(\Omega)}=\sup\operatorname{ess}|f|=\inf\left\{C>0;|f(x)|\leqslant C \text{ a.e } x\in\Omega\right\}.$$

Proof. See for example [2, 3, 5]. □

1.4 Theorem (Hölder's inequality)

Let p and q two conjugate exponents such that $1 \leqslant p, q \leqslant \infty$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, Then $fg \in L^1(\Omega)$ and we have

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

In the particular case where $p = q = 2$, we obtain ***Cauchy-Schwarz inequality***

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Proof. See [3, 5] □

The Dominated Convergence Theorem states that

$$\int \lim_{n \to +\infty} f_n = \lim_{n \to +\infty} \int f_n,$$

when $(f_n)_{n \in \mathbb{N}}$ is a sequence that is simply convergent of integrable functions dominated by a positively integrable function g in the following sense

$$|f_n| \leqslant g, \quad \forall n \in \mathbb{N}.$$

The fact that it is sufficient to have a simple convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ to f represents significant progress compared to statements encountered in the context of the Riemann integral. In general, the Dominated Convergence Theorem is of great practical utility.

1.5 Theorem (Dominated Convergence)

Let $(f_n)_{n \in \mathbb{N}}$ a sequence of $L^1(\Omega)$, such that

1) *$(f_n)_{n \in \mathbb{N}}$ converges almost everywhere in Ω to f.*

2) *There exists a function g from $L^1(\Omega)$ such that $|f_n(x)| \leqslant g(x)$ almost everywhere in Ω for all $n \in \mathbb{N}$.*

Then $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^1(\Omega)$, i.e

$$\lim_{n\to+\infty}\int_\Omega |f_n(x)|\,\mathrm{d}x = \int_\Omega |f(x)|\,\mathrm{d}x.$$

Proof. See [12]. □

Example 1.2.

Let's determine the limit as n tends to infinity of the sequence $(u_n)_{n\geqslant 1}$ defined by

$$u_n = \int_0^{\sqrt{n}} \left(1-\frac{t^2}{n}\right)^n \mathrm{d}t.$$

For all $n \geqslant 1$, we have

$$u_n = \int_{\mathbb{R}} \chi_{[0,\sqrt{n}]}(t)\left(1-\frac{t^2}{n}\right)^n \mathrm{d}t.$$

Let's set

$$\forall t\in\mathbb{R}, f_n(t) = \chi_{[0,\sqrt{n}]}(t)\left(1-\frac{t^2}{n}\right)^n,$$

then for all fixed $t\in\mathbb{R}$, $(f_n)_{n\in\mathbb{N}}$ tends to the function f defined by

$$\mathrm{e}^{-t^2}\chi_{[0,+\infty[}(t),$$

Furthermore, for all $n \geqslant 1$ and $t\in\mathbb{R}$

$$1-\frac{t^2}{n} \leqslant e^{-\frac{t^2}{n}},$$

so,

$$|f_n(t)| \leqslant \mathrm{e}^{-t^2}, \forall t\in\mathbb{R}, \forall n\geqslant 1.$$

The function e^{-t^2} is independent of n and integrable on $\mathbb{R}$. We can apply the Dominated Convergence Theorem to $(f_n)_{n\geqslant 1}$ to obtain

$$\lim_{n\to\infty} u_n = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(t)\,\mathrm{d}t = \int_0^{+\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

1.6 Theorem (Dominated Convergence in $L^p(\Omega)$)

Let $1 \leqslant p < +\infty$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of $L^p(\Omega)$. Assume that

i) $f_n \to f$ almost everywhere in Ω.

ii) There exists a function $g \in L^p(\Omega)$ such that for all n, $\|f_n\|_{L^p(\Omega)} \leqslant g$ almost everywhere in Ω.

Then $f_n \to f$ in $L^p(\Omega)$, i.e $\|f_n - f\|_{L^p(\Omega)} \to 0$ when $n \to +\infty$.

Exercise 1.1.

1) *Prove Theorem 1.2 using the sequence of functions $h_n = |f_n - f|^p$.*

2) *By considering the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $\mathbb{R}$ by*

$$f_n(x) = \begin{cases} nx & \text{if } |x| \leqslant \frac{1}{n}, \\ 0 & \text{if } x > \frac{1}{n}. \end{cases}$$

Show that the Dominated Convergence Theorem is not true in $L^\infty(\mathbb{R})$.

1.3 Definition (Support of a Function)

*Let φ a real function defined on an open set $\Omega \subset \mathbb{R}^N$. The **support** of φ is the set denoted by $\operatorname{supp}(\varphi)$ and is defined as*

$$\operatorname{supp}(\varphi) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$$

The support of φ is then the smallest closed set in $\mathbb{R}^N$ outside of which the function φ is zero.

Example 1.3.

1. *Let's consider the **indicator function** on the interval $[0, 1]$, denoted as*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

The support of this function is the closed interval $[0,1]$, *as it is nonzero only within this interval.*

2. *The* ***Gaussian function*** f *defined by*

$$f(x) = e^{-x^2},$$

has its support as the entire real number line $-\infty < x < +\infty$ *as it's nonzero everywhere.*

3. *Consider the function* f *defined as follows*

$$f(x) = \begin{cases} \sin(x) & \text{if } 0 \leqslant x \leqslant \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The support of $f(x)$ *is* $[0,\pi]$.

Example 1.4.

The support of a function is a fundamental concept, especially in analysis and integration theory, as it helps determine where a function has significant contributions or where it is essential for integration and other operations.

1.7 Theorem (Dual of $L^p(\Omega)$)

Let $p \in [1,\infty[$ *and* q *are Hölder conjugates. Let* T *defined as follows*

$$\begin{aligned} T: \quad L^p(\Omega) &\longrightarrow \mathbb{R} \\ f &\longmapsto T(f). \end{aligned}$$

be a linear continuous form. Then there exists a unique function $g \in L^q(\Omega)$ *such that*

$$T(f) = \int_\Omega g f \,\mathrm{d}x, \quad \forall f \in L^p(\Omega).$$

Furthermore, we have

$$\|g\|_{L^q(\Omega)} = \|T(f)\|_{(L^p(\Omega))'} = \sup_{\substack{f \in L^p(\Omega) \\ f \neq 0}} \frac{|T(f)|}{\|f\|_{L^p(\Omega)}}.$$

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We will now recall a density result for the L^p spaces. Let $C_c(\Omega)$ be the set of functions $\varphi : \Omega \longrightarrow \mathbb{R}$ that are continuous with compact support, which means there exists a compact set $K \subset \Omega$ such that $\varphi = 0$ on $\Omega \backslash K$.

1.8 Theorem (of density)

$C_c(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty[$, i.e., for all $f \in L^p(\Omega)$ and every $\varepsilon > 0$, there exists $\varphi \in C_c(\Omega)$ such that $\|f - \varphi\| < \varepsilon$, or in other words, there exists a sequence $(\varphi_n)n \in \mathbb{N}$ in $C_c(\Omega)$ such that $\|f - \varphi_n\|_{L^p(\Omega)}$ tends to 0 as $n \to +\infty$.

Proof. See [2, 3, 5]. □

Remark 1.2.

$C_c(\Omega)$ is not dense in $L^\infty(\Omega)$. In fact, let $f(x) = 1$ on Ω. Then for any function $\varphi \in C_c(\Omega)$, by setting $K = \operatorname{supp} \varphi$, we have

$$\|f - \varphi\|_{L^\infty(\Omega)} \geqslant \sup_{x \in \Omega \backslash K} |f(x) - \varphi(x)| = \sup_{x \in \Omega \backslash K} |f(x)| = 1.$$

1.4 Definition

Let $1 \leqslant p \leqslant \infty$. $L^p_{\text{loc}}(\Omega)$ It is the set of measurable functions f such that for all compact sets $K \subset \Omega$, $f\chi_K \in L^p(\Omega)$. The function χ_K is the indicator function of K defined by

$$\chi_K(x) = \begin{cases} 1 & \text{if} \quad x \in K, \\ 0 & \text{if} \quad x \notin K. \end{cases}$$

Remark 1.3.

$L^p_{\text{loc}}(\Omega) \subset L^q_{\text{loc}}(\Omega)$, pour $1 \leqslant p \leqslant q \leqslant \infty$.

1.5 Definition (Convolution Product)

Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ where $1 \leqslant p, q \leqslant \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} - 1 \geqslant 0$.

The convolution product of f and g is the function defined on $\mathbb{R}^N$ by

$$f \star g(x) = \int_{\mathbb{R}^N} f(y)g(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^N} f(x-y)g(y)\,\mathrm{d}y.$$

1.9 Theorem (Young's Inequality)

Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ where $1 \leqslant p, q \leqslant \infty$ Then $f \star g \in L^r(\mathbb{R}^N)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and

$$\|f \star g\|_{L^r(\mathbb{R}^N)} \leqslant \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

1.3 Space of Test Functions

1.6 Definition (Space of Test Functions)

A function $\varphi : \Omega \to \mathbb{C}$ is a ***test function*** *if it satisfies the following properties:*

1) *The support of φ, denoted as $\operatorname{supp}\varphi$, is compact.*

2) *φ is of class C^∞ on Ω.*

The ***space of test functions*** *in Ω is denoted as $\mathcal{D}(\Omega)$.*

Let K be a compact subset of Ω. We denote $\mathcal{D}_K(\Omega)$ as the space defined by:

$$\mathcal{D}_K(\Omega) = \{\varphi \in \mathcal{D}(\Omega) \mid \operatorname{supp}\ \varphi \subset K\}.$$

Remark 1.4. *The space $\mathcal{D}(\Omega)$ is not limited to the zero function; otherwise, there would be no purpose in constructing an empty space.*

Example 1.5. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined as follows:*

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } x \in (-1,1), \\ 0 & \text{if } x \notin (-1,1). \end{cases}$$

The function ψ belongs to $\mathcal{D}(\mathbb{R})$ (See exercise 1). If we replace x on the right-hand side with $\|x\| = \sqrt{x_1^2 + \cdots + x_N^2}$, we obtain the function:

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geqslant 1. \end{cases}$$

φ is defined on $\mathbb{R}^N$ and belongs to $\mathcal{D}(\mathbb{R}^N)$.

1.1 Property

1− *Functions in $\mathcal{D}$ and their derivatives are bounded and integrable on Ω.*

2− *Equipped with the usual operations, $\mathcal{D}$ forms a vector space over Ω.*

3− *The product of a C^∞ class function by a function from $\mathcal{D}$ is a function in $\mathcal{D}$.*

4− *The derivatives of a function from $\mathcal{D}$ are also functions in $\mathcal{D}$.*

5− *When $\Omega = \mathbb{R}^N$, for a and λ in $\mathbb{R}^N$, we define the translation $\tau_a\varphi$ and the dilation $\pi_\lambda\varphi$ of the function φ as follows:*

$$\tau_a\varphi(x) = \varphi(x-a), \quad x \in \mathbb{R}^N,$$

and

$$\pi_\lambda\varphi(x) = \varphi\left(\frac{x}{\lambda}\right), \quad x \in \mathbb{R}^N.$$

If φ is a function from $\mathcal{D}$, then $\tau_a\varphi$ and $\pi_\lambda\varphi$ are also in $\mathcal{D}$.

1.3.1 Convergence in $\mathcal{D}(\Omega)$

$\mathcal{D}(\Omega)$ is not a Banach space; it is a Fréchet space (locally convex, metrizable, and complete). In the context of this course, we only need to consider the concept of convergence in this space.

1.7 Definition (Convergence in $\mathcal{D}(\Omega)$)

*A sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ **converges** to φ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that*

1) $\varphi_n \in \mathcal{D}_K(\Omega)$ *for all* $n \in \mathbb{N}$ *and* $\varphi \in \mathcal{D}_K(\Omega)$.

2) $D^\alpha \varphi_n$ *converges uniformly to* $D^\alpha \varphi$ *on* K *for all* $\alpha \in \mathbb{N}^N$, *i.e.,*

$$\forall \alpha \in \mathbb{N}^N, \quad \lim_{n\to+\infty} \sup_{x\in K} |D^\alpha \varphi_n(x) - D^\alpha \varphi(x)| = 0. \tag{1.2}$$

Remark 1.5. *In the case* $N = 1$, $\Omega \subset \mathbb{R}$, (1.2) *becomes*

$$\forall k \in \mathbb{N}, \quad \lim_{n\to+\infty} \sup_{x\in K} \left|\varphi_n^{(k)}(x) - \varphi^{(k)}(x)\right| = 0.$$

Example 1.6. *Let* $f \in \mathcal{D}(\mathbb{R})$, *and for* $n \in \mathbb{N}^\star$, *the sequence* $(\varphi_n)_n$ *is defined by*

$$\forall x \in \mathbb{R}, \quad \varphi_n(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

To study the convergence of φ_n *in* $\mathcal{D}(\mathbb{R})$, *let's first examine pointwise convergence on* $\mathbb{R}$. *Consider a fixed* $x \in \mathbb{R}$:

$$\lim_{n\to+\infty} \varphi_n(x) = \lim_{n\to+\infty} f\left(x + \frac{1}{n}\right) - f(x) = 0.$$

Then, $(\varphi_n)_n$ *converges pointwise to* 0 *on* $\mathbb{R}$. *As* $f \in \mathcal{D}(\mathbb{R})$, *there exist* a *and* $b \in \mathbb{R}$ *such that* $\operatorname{supp}\ f \subset [a, b]$, *so* $\operatorname{supp}\ \varphi_n \subset [a - 1, b]$.

By the Mean Value Theorem, for any $k \in \mathbb{N}$ *and* $x \in \mathbb{R}$, *we get*

$$\left|\varphi_n^{(k)}(x) - \varphi^{(k)}(x)\right| = \left|f^{(k)}\left(x + \frac{1}{n}\right) - f^{(k)}(x)\right| \leqslant \frac{1}{n}\sup_{x\in\mathbb{R}} \left|f^{(k+1)}(x)\right|,$$

So,

$$\lim_{n\to+\infty} \left|\varphi_n^{(k)}(x) - \varphi^{(k)}(x)\right| = 0.$$

$(\varphi_n^{(k)})_n$ *converges uniformly to* $\varphi^{(k)}$ *on* $\mathbb{R}$ *for all* $k \in \mathbb{N}$. *Then,* $(\varphi_n)_n$ *converges simply to* φ *in* $\mathcal{D}(\mathbb{R})$.

1.3.2 Regularization

1.8 Definition (Regularizing Sequence)

*$(\rho_n)_{n\in\mathbb{N}}$ from $\mathbb{R}^N$ is a **regularizing sequence** if, for all $n \in \mathbb{N}$:*

1) $\rho_n \geqslant 0$,

2) $\displaystyle\int_{\mathbb{R}^N} \rho_n(x)\,\mathrm{d}x = 1$,

3) $\operatorname{supp}\ \rho_n \subset \overline{B}(0,\varepsilon_n)$ *with* $\displaystyle\lim_{n\to+\infty} \varepsilon_n = 0$.

Remark 1.6. *Using the function φ from example 1.5, we can construct an infinity of test functions. For a sequence ε_n that is strictly positive and tends to 0 (for example, $\varepsilon_n = \dfrac{1}{n+1}$), we define*

$$\rho_n(x) = \frac{\varphi\left(\dfrac{x}{\varepsilon_n}\right)}{\displaystyle\int_{\mathbb{R}^N} \varphi\left(\dfrac{x}{\varepsilon_n}\right)\,\mathrm{d}x}.$$

One can easily verify that ρ_n satisfies the conditions of Definition 1.3.2.

Exercise 1.2.

1− *Verify that $(\rho_n)_{n\in\mathbb{N}}$ converges almost everywhere to 0.*

2− *Is the sequence $(\rho_n)_{n\in\mathbb{N}}$ convergent in $L^1(\mathbb{R}^N)$?*

1.9 Definition (Locally Integrable Function)

*A function f defined on Ω is called a **locally integrable** function on Ω if f is integrable on each compact subset $K \subset \Omega$.*

The set of locally integrable functions on Ω is denoted by $L^1_{\text{loc}}(\Omega)$.

So, f is locally integrable on Ω if, for all compact subsets $K \subset \Omega$, the product $f\chi_K$ is integrable on Ω, where χ_K is the characteristic function of K, which is equal to 1 on K and 0 outside of K.

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1.10 Definition (Convolution)

et $f \in L^1_{loc}(\mathbb{R}^N)$. *The function*

$$f_n(x) = \int\limits_{\mathbb{R}^N} f(x-y)\rho_n(y)\,\mathrm{d}y = \int\limits_{\mathbb{R}^N} f(x)\rho_n(x-y)\,\mathrm{d}y,$$

It is called ***the convolution of*** f ***by*** ρ_n, *denoted by* $f * \rho_n$ or $\rho_n * f$.

1.10 Theorem (Approximation by Regularization)

Let $(\rho_n)_{n\in\mathbb{N}}$ *be a given regularizing sequence, and let* f *be a locally integrable function on* Ω. *Then the sequence* $f_n = \rho_n * f$ *satisfies the following statements:*

1) $f_n \in \mathcal{D}(\Omega)$.

2) *If* $f \in \mathcal{D}(\Omega)$, *then* $f_n \xrightarrow{\mathcal{D}(\Omega)} f$.

3) *If* $f \in \mathcal{D}$, *then* $f_n \xrightarrow{\mathcal{D}} f$.

4) *If* $f \in L^p(\mathbb{R}^N)$ *and* $1 \leqslant p < +\infty$, *then* $f_n \xrightarrow{L^p(\mathbb{R}^N)} f$.

The following lemma shows that the open set Ω can be expressed as a countable increasing union of compacts.

1.1 Lemma (Exhaustive Sequence)

There exists a family $(K_i)_{i\in\mathbb{N}}$ *of compacts in* Ω *such that:*

1) *For all* $i \geqslant 1$, $K_i \subset \overset{\circ}{K}_{i+1}$.

2) $\Omega = \bigcup\limits_{i=1}^{+\infty} K_i$.

3) *For all compact* K *in* Ω, *there exists* $i_0 \geqslant 1$ *such that* $K \subset K_{i_0}$.

$(K_i)_{i\in\mathbb{N}}$ *is called an* ***exhaustive sequence***.

Proof.

1) $\star$ If $\Omega = \mathbb{R}^N$, it is enough to take $K_i = \overline{B}(0,i)$.

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⋆ If $\Omega \neq \mathbb{R}^N$, let's define for $j \in \mathbb{N}^\star$,

$$A_i = \left\{ x \in \Omega, d(x, C_{\mathbb{R}^N}\Omega) > \frac{1}{i} \right\}, \text{ and } \Omega_i = B(0,i) \cap A_i.$$

It is noticed that

a) A_i is an open set.

b) $\overline{A_i} \subset A_{i+1}$ and $\overline{B}(0,i) \subset B(0,i+1)$, therefore $\overline{\Omega_i} \subset \Omega_{i+1}$.

c) $\overline{\Omega_i} = \overline{B(0,i) \cap A_i} \subset \overline{B(0,i)} \cap \overline{A_i} \subset B(0,i+1) \cap A_{i+1} = \Omega_{i+1}$. Thus $\overline{\Omega_i} \subset \Omega_{i+1} = \overset{\circ}{\Omega}_{i+1}$.

d) $\overline{\Omega_i} \subset \overline{B(0,i)}$ Then $\overline{\Omega_i}$ is compact.

It is enough to take $K_i = \overline{\Omega_i}$, thus $K_i \subset \overset{\circ}{K}_{i+1}$.

2) It is clear that $\overset{+\infty}{\underset{i=1}{\cup}} K_i \subset \Omega$. Let $x \in \Omega$, then there exist $i_1, i_2 \in \mathbb{N}$ such that $x \in B(0,i_1)$ and $d(x, C_{\mathbb{R}^N}\Omega) > \dfrac{1}{i_2}$, so $x \in A_{i_2}$. Putting $i_0 = \max\{i_1, i_2\}$, we deduce that

$$x \in B(0,i_0) \cap A_{i_0} \subset \overline{B(0,i_0) \cap A_{i_0}} = K_{i_0}.$$

3) Let K be a compact subset of Ω. Then $K \subset \underset{i\in\mathbb{N}}{\cup} K_i \subset \underset{i\in\mathbb{N}}{\cup} \overset{\circ}{K}_{i+1}$, consequently, one can extract a finite covering such that

$$K \subset \overset{N}{\underset{i=1}{\cup}} K'_i = K_{i_0}.$$

□

1.2 Lemma (Urysohn's Lemma)

If F is a closed set in $\mathbb{R}^N$ and K is a compact subset of $\mathbb{R}^N$ such that $K \cap F = \emptyset$, then there exists $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that:

1) $0 \leqslant \varphi \leqslant 1$.

2) $\varphi = 1$ *in the neighborhood of K (on $V(K)$).*

3) $\varphi = 0$ *in the neighborhood of F (on $V(F)$).*

Proof.

Let $b > a > 0$, consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{\frac{1}{x-b} - \frac{1}{x-a}} & \text{if } a < x < b, \\ 0 & \text{if not.} \end{cases}$$

f is of class C^∞, and the same goes for the function

$$F(x) = \frac{\int_x^b f(t)\,dt}{\int_a^b f(t)\,dt}.$$

Let's observe that

$$F(x) = \begin{cases} 1 & \text{if} \quad x \leqslant a, \\ 0 & \text{if} \quad x \geqslant b. \end{cases}$$

The function φ given by

$$\varphi(x_1, \dots, x_N) = F\left(x_1^2 + \cdots + x_N^2\right),$$

is of class C^∞ on $\mathbb{R}^N$ and equal to 1 for $\|x\|^2 \leqslant a$ and 0 if $\|x\|^2 \geqslant b$.

Let $\widetilde{B} \subset B$ be two distinct balls in $\mathbb{R}^N$. By performing a linear transformation in $\mathbb{R}^N$, we can construct a function φ in $\mathcal{D}$ that is equal to 1 on $\widetilde{B}$ and zero outside of B.

Let K be a compact set. There exists a finite number of balls $B_1, \dots, B_m$ to cover K, and we can arrange them in such a way that $\overset{m}{\underset{i=1}{\cup}} \overline{B_i} \subset K$. We can even arrange it in such a way that the balls $\widetilde{B_1}, \dots, \widetilde{B_m}$, which respectively contain $B_1, \dots, B_m$, also cover K.

Let $\psi_i \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\psi_i = \begin{cases} 1 & \text{if} \quad x \in \widetilde{B_i}, \\ 0 & \text{if} \quad x \in C_{\mathbb{R}^N} \widetilde{B_i}. \end{cases}$$

and

$$\varphi(x) = 1 - \sum_{i=1}^{m} \left(1 - \psi_i(x)\right).$$

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Then $\varphi \in \mathcal{D}(\mathbb{R}^N), 0 \leqslant \varphi \leqslant 1$, and φ equals 1 in the neighborhood of K and $\operatorname{Supp}\varphi \subset K$. □

1.1 Corollary (Truncation Function)

Let K be a compact subset of Ω. Then there exists $\varphi \in \mathcal{D}(\Omega)$ such that:

1) $0 \leqslant \varphi \leqslant 1$.

2) $\varphi = 1$ *on* $V(K)$.

Remark 1.7.

φ is called a truncation function or a truncation function subordinate to the compact set K, and for any function f defined on Ω, the function $f\varphi$ has compact support in Ω, and $\varphi f = f$ in the neighborhood of K.

Proof. We choose $F = C_{\mathbb{R}^N}\Omega$, which is closed, and we apply Urysohn's lemma.

There exists $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on $V(K)$ and $\varphi = 0$ on $V(F)$. We can take an open set. We still need to show that $\varphi \in \mathcal{D}(\Omega)$. Since $C_{\mathbb{R}^N}(\operatorname{supp}\varphi)$ is the largest open set where $\varphi = 0$, then

$$C_{\mathbb{R}^N}\Omega = F \subset V(F) \subset C_{\mathbb{R}^N} \operatorname{supp}\ \varphi,$$

thus

$$\operatorname{supp}\ \varphi \subset \Omega.$$

□

1.1 Proposition (Exhaustive Sequence with Truncation Functions)

There exists an exhaustive sequence $(K_i)_{i\in\mathbb{N}}$ that covers Ω and functions $\varphi_i \in \mathcal{D}(\Omega)$ such that:

1) $0 \leqslant \varphi_i \leqslant 1$.

2) $\varphi_i = 1$ *on* $V(K_i)$.

Proof.

We apply Urysohn's Lemma. There exists $\varphi_i \in \mathcal{D}(\Omega)$ such that $0 \leqslant \varphi_i \leqslant 1$ and $\varphi_i = 1$ on $V(K_i)$. □

1.11 Theorem

Let K be a compact set such that $K \subset \overset{N}{\underset{i=1}{\cup}} \Omega_i$, where Ω_i are open sets. Then there exist compact sets $K_i \subset \Omega_i$ for $i = 1, \cdots, N$ such that $K \subset \overset{N}{\underset{i=1}{\cup}} K_i$.

Proof.

For each $i \in \{1, \cdots, N\}$, there exists an exhaustive sequence $(M_{ij})_{j\in\mathbb{N}}$ such that

$$\Omega_i \subset \underset{j\in\mathbb{N}}{\cup} M_{i,j} \quad \text{for} \quad i = 1, \cdots, N,$$

then

$$K \subset \overset{N}{\underset{i=1}{\cup}} \Omega_i \subset \underset{\substack{j\in\mathbb{N} \\ i=1,\cdots,N}}{\cup} M_{i,j}.$$

Since K is compact and $M_{i,j}$ are open sets, there exist a finite number of $M_{i,j}$ that cover K, denoted as M'_l with $l = 1, \cdots, m$, such that $K \subset \overset{m}{\underset{i=1}{\cup}} M'_l$. □

The following theorem allows us to decompose a function into a sum of functions with small supports. This process reduces the global study of a function to its local study.

1.12 Theorem (Partition of Unity)

Let K be a compact subset of Ω, where $K \subset \overset{N}{\underset{i=1}{\cup}} \Omega_i$ with $(\Omega_i)_{1\leqslant i\leqslant N}$ being a finite family of open sets included in Ω. Then there exist functions $(\varphi_i)_{1\leqslant i\leqslant N}$ such that:

1) $\varphi_i \in \mathcal{D}(\Omega)$.

2) $0 \leqslant \varphi_i \leqslant 1$.

3) $\sum_{i=1}^{N} \varphi_i = 1$ *on* $V(K)$.

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The proof is based on the previous lemma and Urysohn's lemma.

Proof.

According to the previous lemma, there exist compacts $(K_i)_{1\leqslant i\leqslant N}$ such that

$$K_i \subset \Omega_i,\ i = 1,\cdots,N, \quad \text{and} \quad K \subset \bigcup_{i=1}^{N} K_i.$$

Using Urysohn's lemma applied to $K_i \subset \Omega_i$, there exist functions $(\psi_i)_{1\leqslant i\leqslant N}$ such that $\psi_i \in \mathcal{D}(\Omega_i)$, $0 \leqslant \psi_i \leqslant 1$, and $\psi_i = 1$ on $V(K_i) = \Omega_i' \subset \Omega_i$ for all $i = 1,\cdots,N$.

Let $V = \bigcup_{i=1}^{N} \Omega_i'$. For all $x \in V$, $\sum_{i=1}^{N} \psi_i(x) > 0$.

On the other hand, there exists $\theta \in \mathcal{D}(V)$ such that $\theta = 1$ on $V(K)$ and $0 \leqslant \theta \leqslant 1$.

Now, let's define

$$\psi_0(x) = 1 - \theta(x) = \begin{cases} 1 & \text{if} \quad x \in C_\Omega V(K), \\ 0 & \text{if} \quad x \in V(K). \end{cases}$$

For all $i = 1,\cdots,N$, the function

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{j=0}^{N} \psi_j(x)}, \quad i = 1,\cdots,N,$$

belongs to $\mathcal{D}(\Omega)$ and for all $x \in V(K)$, we have

$$\sum_{i=1}^{N} \varphi_i(x) = \sum_{i=1}^{N} \frac{\psi_i}{\sum_{j=0}^{N} \psi_j(x)} = 1.$$

□

1.4 Exercises on Chapter 1

Exercice 1.

1) Show that the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leqslant 0, \end{cases}$$

is of class C^∞ on $\mathbb{R}$.

2) Let the function $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geqslant 1. \end{cases}$$

for $n \in \mathbb{N}^\star$, we define the sequence $(\varphi_n)_{n\in\mathbb{N}}$ as follows

$$\varphi_n(x) = n\frac{\varphi(nx)}{\varphi(0)}, \quad n \in \mathbb{N}.$$

Deduce that φ and φ_n belong to $\mathcal{D}(\mathbb{R}^N)$, where $\|\cdot\|$ denotes the Euclidean norm

Exercice 2.

Are there test functions φ_1 and φ_2 satisfying 1) and 2)? Justify.

1) $\varphi_1 \in \mathcal{D}(\mathbb{R})$ positive and equal to $\alpha \in \mathbb{R}$ in the neighborhood of 0.

2) $\varphi_2 \in \mathcal{D}(]-1,1[)$ where φ_2' is positive on $]-1,1[$.

Exercice 3.

Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ two sequences defined by

$$f_n(x) = \frac{\varphi(x)}{n} \quad \text{and} \quad g_n(x) = \frac{1}{n}\varphi\left(\frac{x}{n}\right).$$

where φ is a test function.

Study the convergence of the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)n \in \mathbb{N}$ in $\mathcal{D}(\mathbb{R})$.

Exercice 4.

1. Let the sequence $(\varphi_n)_{n\in\mathbb{N}}$ be defined by

$$\varphi_n(x) = e^{-n}\varphi(nx),$$

where $\varphi \in \mathcal{D}(\mathbb{R})$. Show the convergence in $\mathcal{D}(\mathbb{R})$.

2. Study the convergence in $\mathcal{D}(\mathbb{R})$ of the sequence $(\psi_n)_{n\in\mathbb{N}^\star}$ defined by

$$\psi_n(x) = (n+k)^{-k}\varphi(nx), \quad k \in \mathbb{N}.$$

Exercice 5.

Let K a compact subset of $\mathbb{R}^N$ and $\varepsilon > 0$. Show that there exists a function $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$ such that

a) $\varphi_\varepsilon \geqslant 0$,

b) $\varphi_\varepsilon(x) = 1$ for all $x \in K$,

c) supp $\varphi_\varepsilon \subset \bigcup_{x\in K} B(x, \varepsilon)$.

Exercice 6.

Let $(K_p)_{1\leqslant p\leqslant n}$ disjoint compacts in $\mathbb{R}^N$, and $(f_p)_{1\leqslant p\leqslant n}$ are functions of class $\mathcal{C}^\infty(\mathbb{R}^N)$. Show that there exists $f \in \mathcal{D}(\mathbb{R}^N)$ such that $f(x) = f_p(x)$ on K_p.

1.5 Correction of the Exercises in Chapter 1

Correction of Exercise 1

1) The function ψ is of class C^∞ on $]-\infty,0[$ because it is zero on $]-\infty,0[$, and it is of class C^∞ on $]0,+\infty[$ as the composition of two C^∞ functions on $]0,+\infty[$ (the exponential function and the inverse function)

   We can easily show by induction that there exist constants $(C_i)_{1\leqslant i\leqslant n}$ such that

   $$\psi^{(n)}(t)=\left(\frac{C_1}{t^{n+1}}+\frac{C_2}{t^{n+2}}+\cdots+\frac{C_n}{t^{2n}}\right)e^{-\frac{1}{t}} \text{ and } \psi^{(n)}(0)=0.$$

   Let

   $$P\left(\frac{1}{t}\right)=\frac{C_1}{t^{n+1}}+\frac{C_2}{t^{n+2}}+\cdots+\frac{C_n}{t^{2n}},$$

   thus

   $$\lim_{t\overset{>}{\to}0}\psi^{(n)}(t)=\lim_{t\overset{>}{\to}0}P\left(\frac{1}{t}\right)e^{-\frac{1}{t}}=0.$$

   On the other hand

   $$\lim_{t\overset{<}{\to}0}\psi^{(n)}(t)=0.$$

   Derivatives of all orders exist, and they are continuous at 0, which implies that $\psi\in C^\infty(\mathbb{R})$.

2) Let $g:\mathbb{R}^N\longrightarrow\mathbb{R}$ be the function defined by $g(x)=1-\|x\|^2$, the function $\varphi=\psi\circ g$ is of class C^∞ on $\mathbb{R}^N$ As the composition of two functions of class C^∞ on $\mathbb{R}^N$, furthermore

   $$\begin{aligned}\text{supp }\varphi&=\overline{\{x\in\Omega,\ \varphi(x)\neq 0\}}\\&=\overline{\{x\in\Omega,\ \|x\|<1\}}\\&=\{x\in\Omega,\ \|x\|\leqslant 1\}\\&=\overline{B}(0,1),\end{aligned}$$

   $\varphi\in\mathcal{D}(\mathbb{R}^N)$.

φ_n is the composition of two functions of class C^∞ on $\mathbb{R}^N$, so it is of class C^∞ on $\mathbb{R}^N$,

$$\begin{aligned}
\text{supp }\ \varphi_n &= \overline{\{x \in \Omega,\ \varphi_n(x) \neq 0\}} \\
&= \overline{\{x \in \Omega,\ \varphi(nx) \neq 0\}} \\
&= \overline{\{x \in \Omega,\ \|nx\| < 1\}} \\
&= \{x \in \Omega,\ \|x\| \leqslant \frac{1}{n}\} \\
&= \overline{B}\left(0, \frac{1}{n}\right).
\end{aligned}$$

Correction of Exercise 2

1) There exists a test function $\psi \in \mathcal{D}(\mathbb{R})$ equal to 1 in the neighborhood of 0. It is enough to take $\varphi = \alpha\psi$ where $\alpha \in \mathbb{R}^+$.

2) There exists no function $\psi \in \mathcal{D}(\mathbb{R})$ where ψ' is positive on $]-1, 1[$, except the zero function, because ψ is increasing and $\psi(-1) = \psi(1) = 0$.

Correction of Exercise 3

Let's study the convergence of the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$.

1) We have supp $f_n =$ supp $\varphi = [-1, 1] = K$.

The sequence $(f_n)n \in \mathbb{N}$ converges pointwise to 0 on $\mathbb{R}$. Let $k \in \mathbb{N}$

$$\begin{aligned}
\lim_{n\to+\infty} \sup_{x\in[-1,1]} |f_n^{(k)}(x) - f^{(k)}(x)| &= \lim_{n\to+\infty} \sup_{x\in[-1,1]} |f_n^{(k)}(x)| \\
&= \lim_{n\to+\infty} \frac{1}{n} \sup_{x\in[-1,1]} |\varphi^{(k)}(x)| \\
&= 0.
\end{aligned}$$

$(f_n)_{n\in\mathbb{N}}$ converges to 0 in $\mathcal{D}(\mathbb{R})$.

2) g_n is of class C^∞ on $\mathbb{R}$ (composition of functions of class C^∞ on $\mathbb{R}$).

On the other hand

$$\begin{aligned}\text{supp } g_n &= \overline{\{x \in \mathbb{R},\ g_n(x) \neq 0\}}\\ &= \overline{\{x \in \mathbb{R},\ \frac{1}{n}\varphi(\frac{x}{n}) \neq 0\}}\\ &= \overline{\{x \in \Omega,\ \|\frac{x}{n}\| < 1\}}\\ &= \{x \in \Omega,\ \|x\| \leqslant n\}\\ &= [-n, n],\end{aligned}$$

We cannot find a fixed compact set K such that supp $g_n \subset K$ for all $n \in \mathbb{N}$ because for sufficiently large n, supp g_n will not be included in K. $(g_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{D}(\mathbb{R})$.

Correction of Exercise 4

1. Let $\varphi \in \mathcal{D}(\mathbb{R})$, Then there exists $a > 0$ such that supp $\varphi \subset [-a, a]$, so, φ is of class C^∞ on $\mathbb{R}$. We have

$$\text{supp } \varphi_n \subset \left[-\frac{a}{n}, \frac{a}{n}\right] \subset [-a, a] = K.$$

Let's study the pointwise convergence of the sequence $(\varphi_n)_{n\in\mathbb{N}}$.
For $x = 0$, $\varphi_n(0) = e^{-n}\varphi(0)$ then $\lim\limits_{n\to+\infty} \varphi_n(0) = 0$.
Let $x \in \mathbb{R}^\star$ be fixed, and φ has compact support. Then, for sufficiently large n, φ is zero. Then

$$\lim_{n\to+\infty} \varphi_n(x) = \lim_{n\to+\infty} e^{-n}\varphi(nx) = 0.$$

$(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to the zero function on K.
For all k in $\mathbb{N}$

$$\begin{aligned}\lim_{n\to+\infty} \sup_{x\in[-a,a]} |\varphi_n^{(k)}(x) - \varphi^{(k)}(x)| &= \lim_{n\to+\infty} \sup_{x\in[-a,a]} |\varphi_n^{(k)}(x)|\\ &= \lim_{n\to+\infty} n^k e^{-n} \sup_{x\in[-a,a]} |\varphi^{(k)}(nx)|\\ &= 0.\end{aligned}$$

$(\varphi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}(\mathbb{R})$.

2. Let $\varphi \in \mathcal{D}(\mathbb{R})$, then there exists $a > 0$ such that supp $\varphi \subset [-a, a]$, thus ψ_n is of class C^∞ on $\mathbb{R}$

$$\text{supp } \psi_n \subset \left[-\frac{a}{n}, \frac{a}{n}\right] \subset [-a, a] = K.$$

$(\psi_n)n \in \mathbb{N}$ converges pointwise to the zero function on K. Furthermore

$$\lim_{n\to+\infty} \sup_{x\in[-a,a]} |\psi_n^{(j)}(x) - \psi^{(j)}(x)| = \lim_{n\to+\infty} \frac{n^j}{(n+1)^k} \sup_{x\in K} |\varphi^{(j)}(nx)| \neq 0, \quad \text{pour} \quad j \geqslant k.$$

$(\psi_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{D}(\mathbb{R})$.

Correction of Exercise 5

Let $\Omega = \underset{x\in K}{\cup} B(x, \varepsilon)$, then $K \subset \underset{x\in K}{\cup} B(x, \varepsilon) = \Omega$, by Urysohn's Lemma there exists $\varphi_\varepsilon \in \mathcal{D}(\Omega)$ such that $0 \leqslant \varphi_\varepsilon \leqslant 1$ and $\varphi = 1$ on $V(K)$.

Correction of Exercise 6

The compacts $(K_p)1 \leqslant p \leqslant n$ are disjoint. Then there exists a family of open sets $(U_p)_{1\leqslant p\leqslant n}$, such that $K_p \subset U_p$ for $p = 1, \cdots, n$, According to Urysohn's Lemma, there exist $\psi_p \in \mathcal{D}(U_p)$ such that $0 \leqslant \psi_p \leqslant 1$ and $\psi_p = 1$ on $V(K_p)$. Taking

$$f(x) = \sum_{j=0}^{n} \psi_j(x) f_j(x),$$

we deduce that $f \in \mathcal{D}(\mathbb{R}^N)$, and $f(x) = f_p(x)$ on K_p.

CHAPTER 2

DISTRIBUTIONS

2.1 Introduction

The theory of distributions was introduced around 1945 by L. Schwartz to address certain limitations of functions. Distributions have very useful properties for studying certain Sobolev spaces and partial differential equations. It will be seen that a distribution is infinitely differentiable, and this theory has allowed for a clear and rigorous synthesis of certain methods practiced by physicists. For example, physicists use the Dirac delta function, denoted as $\delta(x)$, which is zero for $x \neq 0$ and satisfies $\int_{-\infty}^{+\infty} \delta(x), dx = 1$. From there, one can write, for any function φ

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} (\varphi(x) - \varphi(0))\,\delta(x)\,\mathrm{d}x + \varphi(0)\int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = \varphi(0).$$

Certainly, such a function does not exist, but we can define the mapping $\delta : \varphi \mapsto \varphi(0)$, where δ operates, for example, on the set of continuous functions. This forms the basis of the theory of distributions: certain formulas used in physics are justified within this framework.

In the sequel, Ω denotes a non-empty open set in $\mathbb{R}^N$, $N \geqslant 1$.

Chapter Objectives

- *Understanding the concept of distribution.*

- *Understanding that any distribution can be differentiated an arbitrary number of times.*

2.2 Examples and definitions

2.1 Definition (distributions)

We call a distribution on Ω any linear and continuous form $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$.

Notation 2.1. *We denote by $\mathcal{D}'(\Omega)$ the set of distributions on Ω, in the topological dual space of the space $\mathcal{D}(\Omega)$, and we write $T \in \mathcal{D}'(\Omega)$.*

Remark 2.1.

1− *It is often denoted that the value of T at φ from $\mathcal{D}(\Omega)$ by $\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}}$ ($\langle\cdot,\cdot\rangle$ is called the duality bracket), and we say that the distribution T is applied to φ.*

2− *$\mathcal{D}'(\Omega)$ is equipped with a structure of a topological vector space. Thus, if T and S are two distributions, the sum $T + S$ is defined as follows*

$$\langle T + S, \varphi\rangle = \langle T, \varphi\rangle + \langle S, \varphi\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega),$$

and for $\lambda \in \mathbb{K}$, we define λT as

$$\langle \lambda T, \varphi\rangle = \lambda\langle T, \varphi\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

It can be verified that $T+S$ and λS defined in this way are indeed distributions on Ω, and that $\mathcal{D}'(\Omega)$ equipped with these two operations forms a $\mathbb{K}$-vector space.

2.1 Theorem (Continuity criterion)

Let $T : \mathcal{D}(\Omega) \longleftrightarrow \mathbb{C}$ be a linear form. T is ***continuous*** *if and only if for all compact K of Ω, there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$|\langle T, \varphi\rangle| \leqslant C \sup_{|\alpha|\leqslant m, x\in K} |D^\alpha\varphi(x)|\,, \quad \forall\varphi \in \mathcal{D}_K(\Omega). \tag{2.1}$$

Notation 2.2. *We denote*

$$P_{K,m}(\varphi) = \sup_{|\alpha| \leqslant m, x \in K} |D^{\alpha}\varphi(x)| .$$

Remark 2.2.

Observe that $(PK, m(\varphi))_{m\in\mathbb{N}}$ *is increasing.*

Remark 2.3.

1. *The continuity of the distribution* T *can be demonstrated using sequences. If* $\varphi_n \longrightarrow \varphi$ *in* $\mathcal{D}(\Omega)$*, then* $\langle T, \varphi_n\rangle \longrightarrow \langle T, \varphi\rangle$ *in* $\mathbb{C}$*.*

2. *Since* T *is linear, if we can show that* $\varphi_n \longrightarrow 0$ *in* $\mathcal{D}(\Omega)$*, then* $\langle T, \varphi_n\rangle \longrightarrow 0$ *in* $\mathbb{C}$*, and thus,* T *is continuous.*

Proof.

The upper bound (2.1) indeed expresses the continuity of T for a sequence $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathcal{D}(\Omega)$ with their support in the same compact set K. If $D^{\alpha}\varphi_n$ converges to $D^{\alpha}\varphi$ uniformly for all $\alpha \in \mathbb{N}^N$, it readily follows from the upper bound (2.1) that $\langle T, \varphi_n\rangle$ converges to $\langle T, \varphi\rangle$.

The converse is also true and can be proven as an exercise. □

2.2 Definition (Order of a Distribution)

When the integer $m \in \mathbb{N}$ *in relation (2.1) can be chosen independently of* K*, we say that the distribution* T *is of finite order, and the smallest possible value of* m *is called the order of the distribution* T*.*

2.3 Definition (Radon Measure)

All distributions of order 0 *are called* ***Radon measures*** *on* Ω*.*

Example 2.1.

Let $f \in L^1_{loc}(\Omega)$*, we define* T_f *by*

$$\begin{aligned} T_f :\ \mathcal{D}(\Omega) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \langle T_f, \varphi\rangle = \int_{\Omega} f(x)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

T_f is the distribution associated with the function f.

⋆ *The equality holds because we integrate over* $\operatorname{supp}\varphi$.

⋆ *T_f is linear in $\mathcal{D}(\Omega)$.*

⋆ *Let K be a compact subset of Ω and $\varphi \in \mathcal{D}_K(\Omega)$, thus*

$$\begin{aligned} |\langle T_f, \varphi\rangle| &= \left|\int_\Omega f(x)\varphi(x)\,\mathrm{d}x\right| \\ &\leqslant \int_K |f(x)\varphi(x)|\,\mathrm{d}x \\ &\leqslant \sup_{x\in K}|\varphi(x)| \int_K |f(x)|\,\mathrm{d}x \\ &\leqslant C_K P_{K,0}(\varphi), \end{aligned}$$

where $C_K = \int_K |f(x)|\,\mathrm{d}x$.

This proves that T_f defines a distribution of order 0.

2.2 Theorem

$$T_f = 0 \Longleftrightarrow f(x) = 0 \quad \text{a.e in } \Omega.$$

Remark 2.4.

Given two locally integrable functions f and g that are equal almost everywhere, i.e.,

$$f(x) = g(x) \quad a.e \text{ in } \Omega,$$

the associated distributions are equal,

$$T_f = T_g.$$

The converse is also true (see fo example [7]). So, we have the following theorem.

2.1 Proposition (injection of L^1_{loc} into $\mathcal{D}'$)

The application

$$\begin{array}{ccc} L^1_{\text{loc}}(\Omega) & \to & \mathcal{D}'(\Omega) \\ f & \mapsto & T_f, \end{array}$$

is injective.

Remark 2.5. *Oftentimes, we identify a distribution in the form of T_f with the corresponding function f. Moreover, one could say that a distribution $T \in \mathcal{D}'(\Omega)$ is in $L^1_{loc}(\Omega)$ if there exists $f \in L^1_{loc}(\Omega)$ such that $T = T_f$.*

2.4 Definition (Regular Distribution)

A distribution T is called **regular** *if it is associated with a locally integrable function f, i.e., if there exists $f \in L^1_{loc}(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x = \langle T_f, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 2.6. *When the context does not lead to any confusion, it is often much simpler to simply write T_f as f.*

Example 2.2.

The ***constant regular distribution*** *C is defined by*

$$\langle C, \varphi \rangle = C \int_\Omega \varphi(x)\,\mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

In particular, $C = 0$ defines the zero regular distribution

2.5 Definition (Non-Regular Distribution)

Any distribution that is not regular is called ***non-regular****.*

Example 2.3. *(Dirac distribution)*

Let $a \in \Omega$, We call the ***Dirac mass*** *at a, denoted by δ_a, the distribution defined by*

$$\begin{array}{rcl} \delta_a : \ \mathcal{D}(\Omega) & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \langle \delta_a, \varphi \rangle = \varphi(a). \end{array}$$

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⋆ *δ_a is linear.*

⋆ *Let K a compact subset of Ω and $\varphi \in \mathcal{D}_K(\Omega)$, we have*

$$|\langle \delta_a, \varphi\rangle| = |\varphi(a)| \leqslant \sup_{x\in K}|\varphi(x)| = P_{K,0}(\varphi),$$

So, δ_a is a Radon measure on Ω.

The Dirac distribution δ_a is not regular. Suppose the contrary, namely, there exists a function $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ where $\delta a = T_f$. According to the Urysohn's Lemma, applied to $0 \subset B(0,1)$, there exists $\psi \in \mathcal{D}\left(B(0,1)\right)$ with $\mathrm{supp}(\psi) \subseteq B(0,1)$, $0 \leqslant \psi \leqslant 1$, and $\psi(0) = 1$. Let the function φ defined by

$$\varphi(x) = \psi\left(l\left(x-a\right)\right), \quad l \in \mathbb{N},$$

thus

$$\mathrm{supp}\left(\psi\right) \subset B(a, \frac{1}{l}), \psi\left(a\right) = 1,$$

we deduce that

$$1 = \langle \delta_a, \psi\rangle = \int_{\overline{B}(a,\frac{1}{l})} |f(x)||\psi(x)|\,\mathrm{d}x \leqslant \sup_{x\in\Omega}|\psi(x)| \int_{\overline{B}(a,\frac{1}{l})} |f(x)|\,\mathrm{d}x \longrightarrow 0,$$

when $l \to \infty$, which is absurd.

Remark 2.7. *If $a = 0$, we write δ instead of δ_0, therefore $\langle \delta, \varphi\rangle = \varphi(0)$.*

Example 2.4. *(Dipole Distribution)*
Let $\alpha \in \mathbb{N}^N$ and $a \in \mathbb{R}^N$, we call the distribution dipole, the distribution defined by

$$\langle T, \varphi\rangle = D^\alpha\varphi(a), \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^N).$$

⋆ *T is linear.*

⋆ *Let K be a compact subset of Ω and $\varphi \in \mathcal{D}K(\Omega)$, we have*

$$\begin{aligned} |\langle T, \varphi\rangle| &= |D^\alpha\varphi(a)| \\ &\leqslant \sup_{x\in K}|D^\alpha\varphi(a)| \\ &\leqslant CP_{K,m}(\varphi), \end{aligned}$$

where $C = 1$ *and* $m = |\alpha|$. T *is a distribution of order less than or equal to* $|\alpha|$. *It can be shown that the order is equal to* $|\alpha|$.

In the case of $n = 1$, $\alpha = 1$, *and* $a = 0$,

$$\langle T, \varphi \rangle = \varphi'(0).$$

We know that the order is less than or equal to 1. Let $\varphi \in \mathcal{D}(\mathbb{R})$ *such that* $\varphi'(0) = 1$ *and* $\operatorname{supp}\varphi \subset [-a, a]$ *where* $a > 0$. *Let the sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *be defined by*

$$\varphi_n(x) = \varphi(nx), \quad \forall n \in \mathbb{N}.$$

Thus, $\operatorname{supp}\varphi_n \subset [-a, a]$ *for all* $n \in \mathbb{N}$.

Suppose, for the sake of contradiction, that the order is equal to 0. *Then, for all compact* K *in* Ω, *there exists* $C \geqslant 0$ *such that*

$$\forall \varphi \in \mathcal{D}_K(\Omega),\ |\langle T, \varphi \rangle| \leqslant C P_{K,0}(\varphi).$$

By choosing $K = [-a, a]$, *we find that*

$$\forall \varphi_n \in \mathcal{D}_K(\Omega),\ |\langle T, \varphi_n \rangle| \leqslant C P_{K,0}(\varphi_n),$$

so

$$|n\varphi'(0)| \leqslant C \sup_{x\in K} |\varphi(nx)| \leqslant C \sup_{y\in K} |\varphi(y)|,\ \forall n \in \mathbb{N},$$

which implies that $n \leqslant C'$ *for all* $n \in \mathbb{N}$, *which is absurd. Therefore,* T *is a distribution of order* 1.

Example 2.5. *(Cauchy Principal Value)*

We call the ***Cauchy principal value*** $\operatorname{vp}\frac{1}{x}$, *the application defined by*

$$\langle \operatorname{vp}\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

⋆ *Since* φ *is in* $\mathcal{D}(\mathbb{R})$, *then there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$, *then*

$$\langle \operatorname{vp}\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon\to 0} \left\{ \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}\,\mathrm{d}x \right\}. \tag{2.2}$$

To calculate (2.2), *consider the function* ψ *defined as follows*

$$\psi(x) = \int_0^1 \varphi'(tx)\,\mathrm{d}t, \text{ for } x \neq 0 \text{ where } \psi(0) = \varphi'(0),$$

then

$$\psi(x) = \frac{1}{x}\left[\varphi(tx)\right]_0^1 = \frac{\varphi(x)}{x} - \frac{\varphi(0)}{x}.$$

ψ *is continuous on* $\mathbb{R}$, (2.2) *becomes*

$$\lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \lim_{\varepsilon\to 0} \left\{ \int_{-a}^{-\varepsilon} \frac{\varphi(0)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{a} \frac{\varphi(0)}{x}\,\mathrm{d}x + \int_{-a}^{-\varepsilon} \psi(x)\,\mathrm{d}x + \int_{\varepsilon}^{a} \psi(x)\,\mathrm{d}x \right\},$$

as $\displaystyle\int_{-a}^{-\varepsilon} \frac{\varphi(0)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{a} \frac{\varphi(0)}{x}\,\mathrm{d}x = 0$, *therefore*

$$\lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \int_{-a}^{a} \psi(x)\,\mathrm{d}x,$$

$\mathrm{vp}\,\frac{1}{x}$ *is well-defined due to the continuity of* ψ. ψ.

- ⋆ $\mathrm{vp}\,\frac{1}{x}$ *is linear*

- ⋆ *Let* K *a compact subset of* $\mathbb{R}$ *and Let* $\varphi \in \mathcal{D}_K(\mathbb{R})$. *There exists* $a > 0$ *such that* $K \subset [-a, a]$,

$$\begin{aligned}
\left|\langle \mathrm{vp}\,\tfrac{1}{x}, \varphi\rangle\right| &= \left|\int_{-a}^{a} \psi(x)\,\mathrm{d}x\right| \\
&\leqslant \int_{-a}^{a} |\psi(x)|\,\mathrm{d}x \\
&\leqslant \int_{-a}^{a} \left(\int_0^1 |\varphi'(tx)|\,\mathrm{d}t\right)\mathrm{d}x \\
&\leqslant \sup_{y\in K}|\varphi'(y)| \int_{-a}^{a} \left(\int_0^1 1,\,\mathrm{d}t\right)\mathrm{d}x \\
&\leqslant 2aP_{K,1}(\varphi).
\end{aligned}$$

$\mathrm{vp}\,\frac{1}{x}$ *is a distribution of order* $m \leqslant 1$.

Let's show that $\mathrm{vp}\,\frac{1}{x}$ *is a distribution of order* 1*, for this purpose it suffice to show that the order of* $\mathrm{vp}\,\frac{1}{x}$ *is different to* 0*.*

Assume that the order is equal to 0. Let $\varphi \in \mathcal{D}(\mathbb{R})$*, where* $\varphi = 1$ *in the neighborhood of 0, and* φ *is even and positive. In fact, there exists a test function* $\psi = 1$ *in the neighborhood of* $K = 0$*, and* $0 \leqslant \psi \leqslant 1$*. Then, we can define* φ *as follows:*

$$\varphi(x) = \frac{\psi(x) + \psi(-x)}{2}.$$

φ *is a function in* $\mathcal{D}(\mathbb{R})$*, expressed as the sum of two functions in* $\mathcal{D}(\mathbb{R})$*, and* $\varphi = 1$ *in the neighborhood of* $K = 0$*. We define the sequence* $(\varphi_n(x))_{n\in\mathbb{N}^\star}$ *as follows:*

$$\varphi_n(x) = \varphi(x)\arctan(nx),\ \ n \in \mathbb{N}^\star.$$

Let $K = \operatorname{supp}\varphi$*, there exists* $c \geqslant 0$ *such that*

$$\forall \varphi \in \mathcal{D}_K(\mathbb{R}),\ \ \left|\langle \mathrm{vp}\,\frac{1}{x}, \varphi\rangle\right| \leqslant CP_{K,0}(\varphi).$$

For $\varphi = \varphi_n$*, we get*

$$\forall n \in \mathbb{N}^\star,\ \ \left|\langle \mathrm{vp}\,\frac{1}{x}, \varphi_n\rangle\right| \leqslant CP_{K,0}(\varphi_n) = C',$$

then

$$\begin{aligned} \left|\langle \mathrm{vp}\,\tfrac{1}{x}, \varphi_n\rangle\right| &= \lim_{\varepsilon\to 0} \int\limits_{|x|>\varepsilon} \varphi(x)\frac{\arctan(nx)}{x}\,\mathrm{d}x \\ &= 2\lim_{\varepsilon\to 0} \int\limits_{\varepsilon}^{+\infty} \varphi(x)\frac{\arctan(nx)}{x}\,\mathrm{d}x \\ &= 2\int\limits_{0}^{+\infty} \varphi(x)\frac{\arctan(nx)}{x}\,\mathrm{d}x, \end{aligned}$$

as $\varphi = 1$ *on* $[-\alpha, \alpha]$*, we deduce that*

$$2\int\limits_{0}^{\alpha} \frac{\arctan(nx)}{x}\,\mathrm{d}x \leqslant 2\int\limits_{0}^{+\infty} \varphi(x)\frac{\arctan(nx)}{x}\,\mathrm{d}x,$$

furthermore

$$2\int\limits_{0}^{\alpha} \frac{\arctan(nx)}{x}\,\mathrm{d}x \leqslant C',\ \ \forall n \in \mathbb{N}^\star,$$

by setting $nx = y$, *we find*

$$2\int_0^{\alpha n} \frac{\arctan(y)}{y}\,\mathrm{d}y \leqslant C',\ \forall n \in \mathbb{N}^\star,$$

then

$$\lim_{n\to+\infty} \pi\int_0^{\alpha n} \frac{\arctan(y)}{y}\,\mathrm{d}y \leqslant C',\ \forall n \in \mathbb{N}^\star,$$

so

$$2\int_0^{+\infty} \frac{\arctan y}{y}\,\mathrm{d}y \leqslant C',\ \forall n \in \mathbb{N}^\star$$

which contradicts the fact that the left term is infinite. The distribution $\mathrm{vp}\,\frac{1}{x}$ *is of order* 1.

2.3 Derivatives of Distributions

In this section, we provide a generalization of the concept of derivatives for distributions while retaining the classical notion for typical functions.

Consider a function f *that is of class* C^1 *on* $\mathbb{R}$. *Then, both* f *and* f' *belong to the space* $L^1_{loc}(\mathbb{R})$ *and define distributions on* $\mathbb{R}$. *Furthermore,*

$$\langle f', \varphi\rangle = \int_\Omega f'(x)\varphi(x)\,\mathrm{d}x = -\int_\Omega f(x)\varphi'(x)\,\mathrm{d}x = -\langle f, \varphi'\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

This last formula retains its meaning when we replace f with a distribution.

2.6 Definition (Partial Derivative of a Distribution)

Let T belong to $\mathcal{D}'(\Omega)$, the linear form $\frac{\partial T}{\partial x_i}$ is defined as

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

This is a distribution called the i-th **partial derivative** of T.

Remark 2.8.

1. *It is evident that* $\frac{\partial T}{\partial x_i}$ *is a distribution.*

2. *If T is a distribution of a given order m, then $\frac{\partial T}{\partial x_i}$ is of order $m+1$. This can be shown by considering compact sets and derivatives.*

   Indeed, for all compact sets K in Ω and $\varphi \in \mathcal{D}_K(\Omega)$, we have

$$\begin{aligned}\left|\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle\right| &= \left| - \left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle \right| \\ &\leqslant \sup_{|\alpha| \leqslant m, x \in K} \left| D^\alpha \left(\frac{\partial \varphi}{\partial x_i}(x) \right) \right| \\ &\leqslant \sup_{|\beta| \leqslant m+1, x \in K} \left| D^\beta \varphi(x) \right|.\end{aligned}$$

Remark 2.9.

The fact that differentiation is always possible might raise questions. We know that there are locally integrable and even continuous functions that are not differentiable. However, in the case of distributions, the derivatives are themselves distributions and not functions. This means that the definition of $\frac{\partial T}{\partial x_i}$ can be iterated as many times as needed, and we can define, for any multi-index $\alpha \in \mathbb{N}^N$ and $\varphi \in \mathcal{D}(\mathbb{R})$, $D^\alpha T$

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Example 2.6.

*Let's compute the first and second order derivatives of the **Heaviside** function, defined as*

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leqslant 0. \end{cases}$$

$H \in L^1_{loc}(\mathbb{R})$ then $H \in \mathcal{D}'(\mathbb{R})$. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle H, \varphi \rangle = \int_0^{+\infty} \varphi(x)\, \mathrm{d}x,$$

then

$$\begin{aligned}\langle H', \varphi\rangle &= -\langle H, \varphi'\rangle \\ &= -\int_0^{+\infty} \varphi'(x)\,\mathrm{d}x \\ &= -\left[\varphi(x)\right]_0^{+\infty} \\ &= \varphi(0) \\ &= \langle \delta, \varphi\rangle.\end{aligned}$$

$H' \overset{\mathcal{D}'}{=} \delta$*, where* δ *is identified with the Dirac distribution at* 0.

For all $\varphi \in \mathcal{D}(\mathbb{R})$*, we have*

$$\begin{aligned}\langle H'', \varphi\rangle &= (-1)^2\langle H, \varphi''\rangle \\ &= \int_0^{+\infty} \varphi''(x)\,\mathrm{d}x \\ &= \left[\varphi'(x)\right]_0^{+\infty} \\ &= -\varphi'(0) \\ &= -\langle \delta', \varphi\rangle,\end{aligned}$$

so $H'' \overset{\mathcal{D}'}{=} -\delta'$.

Example 2.7.

1) *Let* f *be differentiable function on* $\mathbb{R}$ *and* $f' \in L^1_{loc}(\mathbb{R})$*. Then,* $T_{f'} \overset{\mathcal{D}'}{=} (T_f)'$*. Indeed, for all* $\varphi \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned}\langle (T_f)', \varphi\rangle &= -\langle T_f, \varphi'\rangle \\ &= -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,\mathrm{d}x \\ &= -\left[f(x)\varphi(x)\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,\mathrm{d}x \\ &= \langle T_{f'}, \varphi\rangle.\end{aligned}$$

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2) *Let $f \in \mathcal{C}^1(\mathbb{R} \setminus \{a\})$ where $a \in \mathbb{R}$. Assume that the left and right limits of f and f' at a exist, noted respectively*

$$\lim_{x \to a^+} f(x) = f(a^+), \quad \lim_{x \to a^-} f(x) = f(a^-).$$

For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}
\langle (T_f)', \varphi \rangle &= -\langle T_f, \varphi' \rangle \\
&= -\int_{-\infty}^{a} f(x)\varphi'(x)\,\mathrm{d}x - \int_{a}^{+\infty} f(x)\varphi'(x)\,\mathrm{d}x \\
&= -\left[f(x)\varphi(x)\right]_{-\infty}^{a} + \int_{-\infty}^{a} f'(x)\varphi(x)\,\mathrm{d}x \\
&\quad -\left[f(x)\varphi(x)\right]_{a}^{+\infty} + \int_{a}^{+\infty} f'(x)\varphi(x)\,\mathrm{d}x \\
&= -(f\varphi)(a^-) + (f\varphi)(a^+) + \langle T_{f'}, \varphi \rangle \\
&= (f(a^+) - f(a^-))\langle \delta_a, \varphi \rangle + \langle T_{f'}, \varphi \rangle,
\end{aligned}$$

then

$$(T_f)' \overset{\mathcal{D}'}{=} T_{f'} + \big(f(a^+) - f(a^-)\big)\,\delta_a,$$

Exercise 2.1. *Assume that $f \in \mathcal{C}^1(\mathbb{R} \setminus \bigcup_{i=1}^{N} \{a_i\})$ where N is a positive integer. Assume that the left and right limits of f and f' at $(a_i)_{i=1,\cdots,N}$ exist. Show that*

$$(T_f)' = T_{f'} + \sum_{i=1}^{N} \big(f(a_i^+) - f(a_i^-)\big)\,\delta_{a_i}.$$

2.1 Lemma

Let $\varphi \in \mathcal{D}(I)$ where I is a subset of $\mathbb{R}$.

1) *Two conditions are equivalent:*

   a) *φ admits a primitive in $\mathcal{D}(I)$.*

   b) $\displaystyle\int_I \varphi(x)\,\mathrm{d}x = 0.$

2) *If the primitive exists, then it is unique.*

Proof.

1) Let's show that 1) implies 2). Suppose there exists $\psi \in \mathcal{D}(I)$ such that $\psi' = \varphi$. Then,

$$\int_I \varphi(x)\,\mathrm{d}x = \int_I \psi'(x)\,\mathrm{d}x = [\psi(x)]_I = 0. \tag{2.3}$$

Now, suppose that $\int_I \varphi(x)\,\mathrm{d}x = 0$. Let's set $\psi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t$. $\psi \in C^\infty$ and $\psi' = \varphi$.

2) Furthermore, ψ has compact support, from which we conclude the equivalence of the two conditions.

For uniqueness, suppose there exist ψ_1 and ψ_2 in $\mathcal{D}(I)$ such that $\psi_1' = \psi_2' = \varphi$, and let $\psi = \psi_1 - \psi_2$. Then, $\psi' = 0$, and therefore $\psi = c$, and since ψ is in $\mathcal{D}(I)$, we have $c = 0$. □

2.3 Theorem

Let I be a subset of $\mathbb{R}$.

1− *Let $T \in \mathcal{D}'(I)$ such that $T' = 0$, then $T = c$.*

2− *For all $T \in \mathcal{D}'(I)$, there exists $S \in \mathcal{D}'(I)$ such that $S' = T$.*

Proof.

1− Let $\theta \in \mathcal{D}(I)$ such that $\int_I \theta(x)\,\mathrm{d}x = 1$. For all $\varphi \in \mathcal{D}(I)$, we define

$$\rho(x) = \varphi(x) - \theta(x) \int_I \varphi(x)\,\mathrm{d}x.$$

Then, $\int_I \rho(x)\,\mathrm{d}x = 0$. We observe that ρ belongs to $\mathcal{D}(I)$. Thanks to the previous lemma, there exists $\psi \in \mathcal{D}(I)$ unique, such that $\psi' = \rho$. Therefore,

$$\varphi(x) = \psi'(x) + \theta(x) \int_I \varphi(x)\,\mathrm{d}x.$$

Assume that $T \in \mathcal{D}'(I)$ such that $T' = 0$, then

$$\begin{aligned}
\langle T, \varphi \rangle &= \langle T, \psi' + \theta \int_I \varphi(x)\, \mathrm{d}x \rangle \\
&= \langle T, \psi' \rangle + c \int_I \varphi(x)\, \mathrm{d}x \\
&= \langle T', \psi \rangle + \langle c, \varphi \rangle \\
&= \langle c, \varphi \rangle,
\end{aligned}$$

where $c = \langle T, \theta \rangle$. Then, $T = c$.

2− Let $T \in \mathcal{D}'(I)$. We are looking for $S \in \mathcal{D}'(I)$ such that $S' = T$.

For all $\varphi \in \mathcal{D}(I)$, we define

$$\langle S, \varphi \rangle = -\langle T, \psi \rangle,$$

where ψ is the unique function in $\mathcal{D}(I)$ defined by the relation

$$\psi'(x) = \varphi(x) - \theta(x) \int_I \varphi(x)\, \mathrm{d}x.$$

T is linear and continuous. Let $K \subset I$ be a compact set, and $\varphi \in \mathcal{D}_K(I)$. Then, there exists m such that (knowing that $K' = K \cup \operatorname{supp} \theta$)

$$\begin{aligned}
|\langle S, \varphi \rangle| &= |\langle T, \psi \rangle| \\
&\leqslant C P_{K',m}(\psi) \\
&\leqslant C max \left(\sup_{x \in K'} |\psi|, \sup_{\substack{x \in K' \\ 1 \leqslant j \leqslant m}} |\psi^{(j)}| \right) \\
&\leqslant C' max \left(\sup_{x \in K'} |\psi|, \sup_{x \in K'} |\varphi| \right),
\end{aligned}$$

where C and C' Are two constants, then there exists C'' such that

$$|\langle S, \varphi \rangle| \leqslant C'' \sup_{\substack{x \in K' \\ 1 \leqslant j \leqslant m}} |\varphi^{(j)}|.$$

Then, S is continuous; hence, $S \in \mathcal{D}'(I)$.

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To verify that $S' = T$, we consider $\varphi \in \mathcal{D}(I)$. We notice that the function ψ associated with φ' according to the relation is φ because

$$\psi''(x) = \varphi'(x) - \theta(x) \int_I \varphi'(x), dx.$$

Since $\varphi' = \psi'$, it follows that $\varphi = \psi$ (thanks to the uniqueness of ψ. Thus,

$$\langle S', \varphi \rangle = -\langle S, \varphi' \rangle = \langle T, \varphi \rangle.$$

Therefore, $S' = T$.

□

2.4 Exercises Chapter 2

Exercice 7.

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Show that only applications 3), 4), and 6) determine distributions.

1) $\langle T_1, \varphi \rangle = |\varphi(0)|$.

2) $\langle T_2, \varphi \rangle = \int\limits_{-\infty}^{+\infty} |\varphi(x)| \, \mathrm{d}x$.

3) $\langle T_3, \varphi \rangle = \int\limits_{-\infty}^{+\infty} \varphi^{(j)}(x) \, \mathrm{d}x$, such that $j \in \mathbb{N}$ is fixed.

4) $\langle T_4, \varphi \rangle = \lim\limits_{n \to +\infty} \left\{ \sum\limits_{k=1}^{n} \varphi(\frac{1}{k}) - n\varphi(0) - \varphi'(0) \ln n \right\}$.

5) $\langle T_5, \varphi \rangle = \sum\limits_{k=1}^{+\infty} \varphi^{(k)}(0)$.

6) $\langle T_6, \varphi \rangle = \sum\limits_{n=0}^{+\infty} \varphi^{(n)}(n)$.

Exercice 8.

For $\varphi \in \mathcal{D}(\mathbb{R})$, we define

$$\langle T, \varphi\rangle = \sum_{k=1}^{+\infty} \frac{1}{k}\left(\varphi(\frac{1}{k}) - \varphi(0)\right).$$

1− Show that T is a distribution of order less than or equal to 1.

2− Prove that T is not of order 0. For this purpose, we recall that if $a < b < c < d$ are real numbers, there exists $\varphi \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $[b, c]$, and the support of φ is contained in $[a, d]$.

Exercice 9.

Calculate the first distributional derivative of the following functions.

1) $f(x) = \operatorname{sgn}(x) = \frac{|x|}{x}$ (sign function) for all $x \in \mathbb{R}^\star$.

2) $g(x) = |x|$ for all $x \in \mathbb{R}$.

Exercice 10.

Let f be the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{2} & \text{if} \quad x \in]-\infty, \frac{\pi}{2}[, \\ 0 & \text{if} \quad x \in [\frac{\pi}{2}, +\infty[. \end{cases}$$

1) Calculate the first and second distributional derivatives of f.

2) Deduce that f is a solution of a differential equation.

Exercice 11.

Calculate the first derivative of $f(x) = \ln|x|$, $x \in \mathbb{R}^\star$ in the sense of distributions.

Exercice 12.

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Show that the following linear form is a distribution.

$$< \operatorname{Fp}\left(\frac{1}{x^2}\right) \varphi\rangle = \lim_{\varepsilon \to 0^+}\left(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\, \mathrm{d}x - 2\frac{\varphi(0)}{\varepsilon}\right)$$

Find a connection between $\operatorname{vp}\left(\frac{1}{x}\right)$ and $\operatorname{Fp}\left(\frac{1}{x^2}\right)$.

2.5 Correction of the Exercises in Chapter 2

Correction of Exercise 7

1) *The applications* T_1 *and* T_2 *do not define distributions because they are not linear.*

3) *Let* $j \in \mathbb{N}$ *be fixed, we have*

$$\begin{aligned} T_3 : \quad \mathcal{D}(\Omega) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \langle T_3, \varphi \rangle = \int_{-\infty}^{+\infty} \varphi^{(j)}(x)\,\mathrm{d}x. \end{aligned}$$

Let $\varphi \in \mathcal{D}(\Omega)$*, for all* $j \in \mathbb{N}$*,* $\varphi^{(j)} \in \mathcal{D}(\Omega)$ *and* $\operatorname{supp} \varphi^{(j)} \subset \operatorname{supp} \varphi$*, thus*

$$\int_{-\infty}^{+\infty} \varphi^{(j)}(x)\,\mathrm{d}x = \int_{\operatorname{supp} \varphi} \varphi^{(j)}(x)\,\mathrm{d}x.$$

- ⋆ *The integral exists because it is the integral of a continuous function over the compact set* $\operatorname{supp} \varphi$*.*
- ⋆ T_3 *is linear on* $\mathcal{D}(\Omega)$*.*
- ⋆ *Let* K *be a compact subset of* Ω *and* $\varphi \in \mathcal{D}_K(\Omega)$*, we have*

$$\begin{aligned} |\langle T_3, \varphi \rangle| &= \left| \int_{\Omega} \varphi^{(j)}(x)\,\mathrm{d}x \right| \\ &= \left| \int_{K} \varphi^{(j)}(x)\,\mathrm{d}x \right| \\ &\leqslant \sup_{x \in K} |\varphi^{(j)}(x)| \int_{K} 1\,\mathrm{d}x \\ &\leqslant C_K P_{K,j}(\varphi), \end{aligned}$$

where $C_K = \operatorname{mes}(K)$*.*

T_3 *is a distribution of order less than or equal to* j*.*

4) *Let* φ *in* $\mathcal{D}(\mathbb{R})$ *then there exists* $N > 0$ *such that* $\operatorname{supp} \varphi \subset [-N, N]$

⋆ *First, let's show that T is well-defined. Let's define*

$$u_n = \sum_{k=1}^{n} \varphi(\frac{1}{k}) - n\varphi(0) - \varphi'(0)\ln n. \tag{2.4}$$

To calculate $\lim_{n\to+\infty} u_n$, *we use the Taylor expansion to order* 2 *in the neighborhood of* 0

$$\varphi\left(\frac{1}{k}\right) = \varphi(0) + \frac{1}{k}\varphi'(0) + \frac{1}{2k^2}\varphi''(c_k).$$

Then

$$\begin{aligned} u_n &= \sum_{k=1}^{n} \varphi(0) + \sum_{k=1}^{n} \frac{1}{k}\varphi'(0) + \sum_{k=1}^{n} \frac{1}{k^2}\varphi''(c_k) - n\varphi(0) - \varphi'(0)\ln(n) \\ &= n\varphi(0) + \sum_{k=1}^{n} \frac{1}{k}\varphi'(0) + \sum_{k=1}^{n} \frac{1}{k^2}\varphi''(c_k) - n\varphi(0) - \varphi'(0)\ln(n) \\ &= \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right)\varphi'(0) + \sum_{k=1}^{n} \frac{1}{2k^2}\varphi''(c_k). \end{aligned}$$

Let's define

$$v_n = \sum_{k=1}^{n} \frac{1}{2k^2}\varphi''(c_k), w_n = \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right).$$

We have

$$\left|\frac{1}{2k^2}\varphi''\right| \leqslant \sup_{x\in K} |\varphi''| \frac{1}{2k^2}.$$

$\sum \frac{1}{n^2}$ *is a convergent Riemann series, according to the comparison criterion* v_n *is a sequence associated with a convergent series. Using the comparison criterion with an integral, we can prove that the sequence* v_n *is convergent. So,* T_4 *is well-defined.*

⋆ T_4 *is linear.*

⋆ *Let* K *be a compact subset of* $\mathbb{R}$ *and Let* $\varphi \in \mathcal{D}_K(\mathbb{R})$, *there exists* $a > 0$ *such* *that* $K \subset [-a,a]$*, so*

$$\begin{aligned}
|\langle T_4,\varphi\rangle| &= \lim_{n\to+\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)\varphi'(0)+\sum_{k=1}^{n}\frac{1}{2k^2}\varphi''(c_k)\right]\\
&\leqslant \sup_{x\in K}|\varphi'(x)|\lim_{n\to+\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)+\sup_{x\in K}|\varphi''(x)|\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2k^2}\\
&\leqslant S_1P_{K,1}(\varphi)+S_2P_{K,2}(\varphi)\\
&\leqslant CP_{K,2}(\varphi).
\end{aligned}$$

T_4 *is a distribution of order* $m\leqslant 2$*.*

5) T_5 *is not well-defined because for* $K=\{0\}$*, there exists a function* $\varphi\in\mathcal{D}(\mathbb{R})$ *that satisfies* $\varphi=1$ *in the neighborhood of* K*. By setting*

$$\psi(x)=e^x\varphi(x),$$

we find that there exist constants $(c_i)_{0\leqslant i\leqslant k}$ *such that*

$$\begin{aligned}
\psi^{(k)}(x) &= \sum_{i=0}^{k}c_ie^x\varphi^{(i)}(x)\\
&= \varphi(x)+\sum_{i=1}^{k}c_ie^x\varphi^{(i)}(x),
\end{aligned}$$

the sequence $\psi^{(k)}(x)=\varphi(0)=1$ *does not converge to* 0*, so the series is divergent, and therefore* T_5 *does not define a distribution.*

6) *Let* φ *be in* $\mathcal{D}(\mathbb{R})$*, there exists* $N>0$ *such that* $\operatorname{supp}\varphi\subset[-N,N]$*, then for all* $n>N$*,* $\varphi(n)=0$*. Thanks to the uniqueness of for all* $n>N$*,* $\varphi^{(n)}(n)=0$*, consequently*

$$\langle T_6,\varphi\rangle=\sum_{n=0}^{+\infty}\varphi^{(n)}(n)=\sum_{n=0}^{N}\varphi^{(n)}(n).$$

T_6 *is well-defined because it is a finite sum.* T *is obviously linear. As for its continuity, for all compact* K *in* $\mathbb{R}$ *and* $\varphi\in\mathcal{D}_K(\mathbb{R})$*, there exists* $N>0$ *such that* $K\subset[-N,N]$*, so*

$$|\langle T,\varphi\rangle|\leqslant(N+1)P_{K,N}(\varphi).$$

The distribution T_6 is not of finite order.

Correction of Exercise 8

1− *Let $\varphi \in \mathcal{D}(\mathbb{R})$, the Taylor expansion to order 1 in the neighborhood of 0 gives*

$$\varphi\left(\frac{1}{k}\right) = \varphi(0) + \frac{1}{k}\varphi'(c_k),$$

thus

$$\langle T, \varphi\rangle = \sum_{k=1}^{\infty} \frac{1}{k^2}\varphi'(c_k),$$

we deduce that

$$|\langle T, \varphi\rangle| \leqslant \sup_{x\in\mathbb{R}} |\varphi'(x)| \sum_{k=1}^{\infty} \frac{1}{k^2},$$

the series is absolutely convergent, then T is well-defined.

It is easy to show that T is linear. Let's prove the continuity of T. Let K be a compact subset of $\mathbb{R}$, and $\varphi \in \mathcal{D}_K(\mathbb{R})$, then

$$\begin{aligned}|\langle T, \varphi\rangle| &\leqslant \sup_{x\in\mathbb{R}} |\varphi'(x)| \sum_{k=1}^{\infty} \frac{1}{k^2}\\ &\leqslant CP_{K,1}(\varphi),\end{aligned}$$

T is a distribution of order less than or equal to 1.

2− *Assume that the order of T equals 0. Then, for all compacts K, there exists $C > 0$ such that*

$$\begin{aligned}|\langle T, \varphi\rangle| &\leqslant CP_{K,0}(\varphi)\\ &= C\sup_{x\in\mathbb{R}} |\varphi(x)|,\end{aligned}$$

For all $n \geqslant 1$, there exists a function $\varphi_n \in \mathcal{D}(\mathbb{R})$, satisfying $\operatorname{supp}\varphi_n \subset]0, 2[$, where $\varphi_n = 1$ on $[\frac{1}{n}, 1]$, and $0 \leqslant \varphi_n \leqslant 1$, then

$$\langle T, \varphi_n\rangle = \sum_{k=1}^{\infty} \frac{1}{k}\left(\varphi_n\left(\frac{1}{k}\right) - \varphi_n(0)\right).$$

since $\frac{1}{k} \in [\frac{1}{n}, 1]$, *then* $\varphi_n\left(\frac{1}{k}\right) = 1$ *and* $\varphi_n(0)$, *thus*

$$\langle T, \varphi_n \rangle \geqslant \sum_{k=1}^{n} \frac{1}{k}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ *is a divergent series, then* $\langle T, \varphi_n \rangle \to +\infty$. *On the other hand, for* $K = [0, 2]$, *we find*

$$|\langle T, \varphi_n \rangle| \leqslant C \sup_{x \in \mathbb{R}} |\varphi_n(x)|,$$

a contradiction with the fact that $\langle T, \varphi_n \rangle \to +\infty$.

Correction of Exercise 9

1) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$, *we have*

$$\begin{aligned}
\langle T_f', \varphi \rangle &= -\langle T_f, \varphi' \rangle \\
&= \int_{-\infty}^{+\infty} f(x)\varphi'(x)dx \\
&= \int_{-\infty}^{0} \varphi'(x)dx - \int_{0}^{+\infty} \varphi'(x)dx \\
&= 2\varphi(0) \\
&= 2\langle \delta, \varphi \rangle
\end{aligned}$$

Thus,

$$T_f' = 2\delta.$$

We can obtain this result using the jump formula.

2) *For all* $\varphi \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned}
\langle T_g', \varphi \rangle &= -\langle T_g, \varphi' \rangle \\
&= \int_{-\infty}^{+\infty} |x|\varphi'(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{0} x\varphi'(x)\, \mathrm{d}x - \int_{0}^{+\infty} x\varphi'(x)\, \mathrm{d}x.
\end{aligned}$$

Integration by parts yields

$$\langle T_g', \varphi\rangle = -\int_{-\infty}^{0} \varphi(x)\,\mathrm{d}x + \int_{0}^{+\infty} \varphi(x)\,\mathrm{d}x.$$

Thus, if we denote

$$u(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$, we find

$$\langle T_g', \varphi\rangle = \int_{-\infty}^{+\infty} u(x)\varphi(x)\,\mathrm{d}x = \langle T_u, \varphi\rangle.$$

This means that u is the distributional derivative of the absolute value function.

Correction of Exercise 10

1) *Let's calculate $(T_f)'$ and $(T_f)''$. We have*

$$f'(x) = \begin{cases} \frac{\cos x}{2} & \text{if } \; x \in]-\infty, \frac{\pi}{2}[\\ 0 & \text{if } \; x \in]\frac{\pi}{2}, +\infty[\end{cases}$$

and

$$f''(x) = \begin{cases} \frac{\sin x}{2} & \text{if } \; x \in]-\infty, \frac{\pi}{2}[\\ 0 & \text{if } \; x \in]\frac{\pi}{2}, +\infty[\end{cases}$$

According to the jump theorem

$$(T_f)' = T_{f'} + \big(f(a^+) - f(a^-)\big)\,\delta_a,$$

for $a = \frac{\pi}{2}$, we have

$$\lim_{x\to a^+} f(x) = 0, \;\; \lim_{x\to a^-} f(x) = \frac{1}{2}.$$

so

$$(T_f)' = T_{f'} - \frac{1}{2}\delta_{\frac{\pi}{2}},$$

To calculate the second derivative, using the jump formula twice, we have

$$(T_f)'' = \left(T_{f'} - \frac{1}{2}\delta_{\frac{\pi}{2}}\right)'$$
$$= (T_{f'})' - \frac{1}{2}\delta'_{\frac{\pi}{2}}.$$

as

$$(T'_f)' = T_{f''} + \left(f'(a^+) - f'(a^-)\right)\delta_a,$$

and

$$\lim_{x\to a^+} f'(x) = 0, \ \lim_{x\to a^-} f'(x) = 0.$$

so

$$(T'_f)' = T_{f''}$$

hence

$$(T_f)'' = T_{f''} - \frac{1}{2}\delta'_{\frac{\pi}{2}}.$$

2) *We notice that*

$$T_{f''} = -T_f,$$

so T_f satisfies the differential equation

$$(T_f)'' + T_f = -\frac{1}{2}\delta'_{\frac{\pi}{2}}$$

Correction of Exercise 11

Let $\varphi \in \mathcal{D}(\mathbb{R}^\star)$ and $a > 0$ such that the support of φ is contained in $[-a, a]$. Let T be the distribution associated with the function $\ln|x|$. Then

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$$
$$= -\int_{-a}^{a} \varphi'(x)\ln|x|dx$$
$$= -\lim_{\varepsilon\to 0}\left(\int_{-a}^{-\varepsilon} \varphi'(x)\ln(-x)dx + \int_{\varepsilon}^{a} \varphi'(x)\ln(x)dx\right)$$

We have

$$\int_{\varepsilon}^{a} \varphi'(x)\ln(x)dx = [\varphi(x)\ln(x)]_{\varepsilon}^{a} - \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx$$
$$= -\varphi(\varepsilon)\ln(\varepsilon) - \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx.$$

and

$$\int_{-a}^{-\varepsilon} \varphi'(x)\ln(-x)dx = [\varphi(x)\ln(-x)]_{-a}^{-\varepsilon} - \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx$$
$$= \varphi(-\varepsilon)\ln(\varepsilon) - \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx.$$

Furthermore,

$$\big(\varphi(-\varepsilon) - \varphi(\varepsilon)\big)\ln(\varepsilon) = (\varphi(0) - \varepsilon\varphi'(0) - \varphi(0) - \varepsilon\varphi'(0) + o(\varepsilon))\ln(\varepsilon)$$
$$= -2\varphi'(0)\varepsilon\ln(\varepsilon) + o\big(\varepsilon\ln(\varepsilon)\big).$$

$\varepsilon\ln(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, so*

$$\langle T', \varphi\rangle = \lim_{\varepsilon\to 0}\left(\int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx\right) = \langle \mathrm{vp}\,\frac{1}{x}, \varphi\rangle.$$

Correction of Exercise 12

Let $\varphi \in \mathcal{D}(\mathbb{R})$*, and there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$*. The first-order Taylor formula is written as*

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\varphi''(\xi_x),$$

then

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\,\mathrm{d}x = \int_{|x|>\varepsilon} \frac{\varphi(0)}{x^2}\,\mathrm{d}x + \int_{|x|>\varepsilon} \frac{\varphi'(0)}{x}\,\mathrm{d}x + \int_{-a}^{a} \varphi''(\xi_x)\,\mathrm{d}x.$$

The second integral of the second term on the right-hand side vanishes because we integrate an odd function over a symmetric interval, so

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\,\mathrm{d}x = 2\frac{\varphi(0)}{\varepsilon} + \int_{|x|>\varepsilon} \varphi''(x)\,\mathrm{d}x,$$

thus

$$|\langle T,\varphi\rangle| \leqslant 2a \sup_{x\in[-a,a]} |\varphi''(x)|$$

T *is a distribution of order less than or equal to* 2.

Now, let's calculate the derivative of $\mathrm{vp}(\frac{1}{x})$. *For* $\varphi \in \mathcal{D}(\mathbb{R}^n)$, *using integration by parts, we find*

$$\begin{aligned}\left\langle \mathrm{vp}\left(\frac{1}{x}\right)', \varphi \right\rangle &= -\left\langle \mathrm{vp}\left(\frac{1}{x}\right), \varphi' \right\rangle \\ &= -\lim_{\varepsilon\to 0^+} \int\limits_{|x|>\varepsilon} \frac{\varphi'(x)}{x}\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0^+} \left[\frac{\varphi(x)}{x}\right]_{\varepsilon}^{-\varepsilon} + \int\limits_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\,\mathrm{d}x.\end{aligned}$$

However, using Hadamard's lemma, when $\varepsilon \to 0$,

$$\left[\frac{\varphi(x)}{x}\right]_{\varepsilon}^{-\varepsilon} = -2\frac{\varphi(0)}{\varepsilon} + o(1)$$

thus

$$\left\langle \mathrm{vp}\left(\frac{1}{x}\right)', \varphi \right\rangle = -\left\langle \mathrm{Fp}\left(\frac{1}{x^2}\right), \varphi \right\rangle.$$

CHAPTER 3

SEQUENCES AND SERIES OF DISTRIBUTIONS

3.1 Introduction

Many distributions are defined through the limit of a sequence of regular distributions or as a sum of distributions. In this chapter, we present the concept of convergence of sequences and series of distributions.

Chapter Objectives

- *Get familiar with the concept of convergence of a sequence or series of distributions.*

3.2 Convergence of Distributions

The concept of convergence of distributions is also called weak convergence as follows:

3.1 Definition (Convergence in $\mathcal{D}'(\Omega)$)

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions and $T \in \mathcal{D}'(\Omega)$. We say that $(T_n)_{n\in\mathbb{N}}$ ***converges*** *to T in $\mathcal{D}'(\Omega)$ if and only if for any test function $\varphi \in \mathcal{D}(\Omega)$, the sequence of real numbers $\langle T_n, \varphi\rangle$ converges to $\langle T, \varphi\rangle$, i.e.,*

$$\lim_{n\to+\infty}\langle T_n, \varphi\rangle = \langle T, \varphi\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

3.2 Definition

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of locally integrable functions and (T_{f_n}) be the sequence of associated regular distributions. If the sequence $(T_{f_n})_{n\in\mathbb{N}}$ converges to a distribution T, we say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to T in the sense of distributions, and we write

$$\lim_{n\to\infty} f_n = T \quad \text{ in } \mathcal{D}'\left(\mathbb{R}^n\right).$$

Example 3.1.

1) *Let $f_n(x) = \cos(nx)$, $f_n \in L^1_{\text{loc}}(\mathbb{R})$; therefore, $T_{f_n} \in \mathcal{D}'(\mathbb{R})$. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have*

$$\langle T_{f_n}, \varphi\rangle = \int_{-\infty}^{+\infty} \cos(nx)\varphi(x)\, dx \underset{n\to\infty}{\longrightarrow} 0,$$

then T_{f_n} converges to 0 in $\mathcal{D}'(\mathbb{R})$.

2) *Let $n \in \mathbb{N}$, the Dirac distribution δ_n converges to 0 in $\mathcal{D}'(\mathbb{R})$. Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R})$, there exists $n_0 > 0$ such that $\operatorname{supp}\varphi \subset [-n_0, n_0]$, and we have*

$$\langle \delta_n, \varphi\rangle = \varphi(n) = 0, \quad \forall n \geqslant n_0.$$

3.1 Theorem

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{D}'(\Omega)$. If $T_n \overset{\mathcal{D}'(\Omega)}{\to} T$, then for all $\alpha \in \mathbb{N}^N$, $D^\alpha T_n \overset{\mathcal{D}'(\Omega)}{\to} D^\alpha T$.

Proof.

Let $\varphi \in \mathcal{D}(\mathbb{R})$, and $\alpha \in \mathbb{N}^N$ be a multi-index, we have

$$\langle D^\alpha T_n, \varphi\rangle = (-1)^{|\alpha|}\langle T_n, D^\alpha \varphi\rangle \overset{n\to+\infty}{\longrightarrow} (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle = \langle D^\alpha T, \varphi\rangle.$$

Hence, we get the desired result. □

Remark 3.1.

This property is of fundamental importance. Differentiation becomes an operation that is always defined and continuous, whereas it is well-known that such a property is entirely false for the standard differentiation of functions.

3.1 Proposition

The following conditions are sufficient for $f_n \to f$ in $\mathcal{D}'(\Omega)$.

1) $f_n \to f$ *in* $L^1(\Omega)$.

2) $f_n \to f$ *in* $L^2(\Omega)$.

Proof.

1) Assume that $f_n \to f$ in $L^1(\Omega)$. Then, for all $\varphi \in \mathcal{D}(\Omega)$ and for all $n \in \mathbb{N}$, we have

$$\left|\int_\Omega f_n(x)\varphi(x)\,dx - \int_\Omega f(x)\varphi(x)\,dx\right| \leqslant \int_\Omega |\varphi(x)|\,|f_n(x) - f(x)|\,dx$$
$$\leqslant \sup_{x\in\Omega}|\varphi(x)|\,\|f_n - f\|_{L^1(\Omega)},$$

therefore $f_n \to f$ in $\mathcal{D}'(\Omega)$.

2) Assume that $f_n \to f$ in $L^2(\Omega)$. According to the Cauchy-Schwarz inequality, for all $\varphi \in \mathcal{D}(\Omega)$ and for all $n \in \mathbb{N}$, we have

$$\left| \int_\Omega f_n(x)\varphi(x)\,dx - \int_\Omega f(x)\varphi(x)\,dx \right| \leqslant \int_\Omega |\varphi(x)|\,|f_n(x) - f(x)|\,\varphi(x)\,dx$$
$$\leqslant \|\varphi\|_{L^2(\Omega)} \|f_n - f\|_{L^2(\Omega)},$$

consequently $f_n \to f$ in $\mathcal{D}'(\Omega)$.

□

Remark 3.2.

According to the previous proposition, convergence in L^1 or L^2 implies convergence in $\mathcal{D}'$. In fact, it can be shown that convergence in L^p with $1 \leqslant p \leqslant +\infty$ implies convergence in $\mathcal{D}'$.

3.2 Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $L^1(\mathbb{R}^N)$ such that

1) $\operatorname{supp} f_n \subset B(0,\varepsilon_n)$ *with* $\lim\limits_{n\to+\infty} \varepsilon_n = 0$.

2) *For all* $n \in \mathbb{N}$, $\int_{\mathbb{R}^N} f_n(x)\,dx = 1$.

3) *For all* $n \in \mathbb{N}$, $f_n \geqslant 0$.

then $(f_n)_{n\in\mathbb{N}}$ converges to δ in $\mathcal{D}'(\mathbb{R}^N)$.

Proof.

Let $\varphi \in \mathcal{D}(\mathbb{R})$, given the assumptions, applying the sequence $(f_n)_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned}
|\langle f_n, \varphi\rangle - \langle \delta, \varphi\rangle| &= \left| \int_{\mathbb{R}^N} f_n(x)\varphi(x)\,dx - \varphi(0) \right| \\
&= \left| \int_{B(0,\varepsilon_n)} f_n(x)\left(\varphi(x) - \varphi(0)\right)\,dx \right| \\
&\leqslant \sup_{x\in B(0,\varepsilon_n)} |\varphi(x) - \varphi(0)| \int_{\mathbb{R}^N} f_n(x)\,dx \\
&\leqslant |\varphi(x_n) - \varphi(0)| \overset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

while for all $n \in \mathbb{N}$, there exists $x_n \in B(0, \varepsilon_n)$ such that

$$\sup_{x\in B(0,\varepsilon_n)} |\varphi(x) - \varphi(0)| = |\varphi(x_n) - \varphi(0)|\,.$$

□

Example 3.2.

Let the function f defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

f is a C^∞ function on $\mathbb{R}$, and we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$

Consider the sequence defined by

$$f_n(x) = nf(nx), \quad \forall n \in \mathbb{N}.$$

According to the previous theorem, we have

$$\lim_{n\to\infty} (T_{f_n})' = \lim_{n\to\infty} T_{f_n'} = \delta',$$

as f_n is differentiable on $\mathbb{R}$, and then

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} -\frac{n^3 x}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}} \varphi(x)\,dx = -\varphi'(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

3.2 Proposition (Uniform convergence on any compact set implies convergence in $\mathcal{D}'$)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $\mathbb{R}^N$. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly on any compact set to f, then $(f_n)_{n\in\mathbb{N}}$ converges in the sense of distributions to f.

Proof.

For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$|\langle T_{f_n}, \varphi\rangle - \langle T_f, \varphi\rangle| \leqslant \sup_{x\in K} |f(x) - f_n(x)| \sup_{x\in K} |\varphi(x)|$$

where K is a compact set containing the support of φ. As $(f_n)_{n\in\mathbb{N}}$ converges uniformly on K to f, the right-hand side of the previous inequality tends to 0 as n tends to infinity. □

3.3 Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{\text{loc}}(\Omega)$ such that

1) *$f_n \longrightarrow f$ almost everywhere in Ω*

2) *there exists $g \in L^1_{\text{loc}}(\Omega)$, $|f_n(x)| \leqslant g(x)$ for all $n \in \mathbb{N}$.*

then $f_n \xrightarrow{\mathcal{D}'} f$ and $f \in L^1_{\text{loc}}(\Omega)$.

Proof.

$f_n \xrightarrow{\mathcal{D}'} f$ if and only if

$$\forall \varphi \in \mathcal{D}(\mathbb{R}), \quad \int_\Omega f_n(x)\varphi(x)\,\mathrm{d}x \to \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$, and *supp*$\varphi \subset K$, for all $\varepsilon > 0$, let $K_\varepsilon = \underset{x\in K}{\cup} B(x,\varepsilon)$, we have

$$\varphi_n = f_n\varphi \to f\varphi \quad \text{a.e in } \ \Omega$$

Furthermore

$$|\varphi_n(x)| = |f_n(x)\varphi(x)| \leqslant g(x)|\varphi(x)| \quad \text{for all} \quad n \in \mathbb{N}.$$

$g|\varphi| \in L^1(K_\varepsilon)$, then by Lebesgue's dominated convergence theorem, $f\varphi \in L^1(K_\varepsilon)$ and

$$\forall \varphi \in \mathcal{D}(\mathbb{R}), \quad \int_\Omega f_n(x)\varphi(x)\,\mathrm{d}x \to \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.$$

and then $f_n \xrightarrow{\mathcal{D}'} f$ and $f \in L^1_{\text{loc}}(\Omega)$. □

3.3 Distribution Series

3.3 Definition

We consider the sequence of distributions $(T_n)_{n\in\mathbb{N}}$. We say that the distribution series $\sum\limits_{n\in\mathbb{N}} T_n$ defines a distribution T in $\mathcal{D}'$ if the sequence of partial sums

$$S_n = \sum_{k=0}^{n} T_k$$

converges in the distribution sense to T, meaning

$$\lim_{n\to\infty} S_n \overset{\mathcal{D}'}{=} \lim_{n\to\infty} \sum_{k=0}^{n} T_k \overset{\mathcal{D}'}{=} T.$$

3.4 Theorem (term-by-term differentiation)

If the series $\sum\limits_{n\in\mathbb{N}} T_n$ converges to the distribution S, then the series $\sum\limits_{n\in\mathbb{N}} D^\alpha T_n$ converges to $D^\alpha S$, that is

$$S = \sum_{n\in\mathbb{N}} T_n \longrightarrow D^\alpha S = \sum_{n\in\mathbb{N}} D^\alpha T_n.$$

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3.4 Exercises on Chapter 3

Exercice 13.

Let the sequence $(f_n)_{n\in\mathbb{N}}$ be defined as

$$f_n(x) = \begin{cases} n & \text{if} \quad x \in]-\frac{1}{2n}, \frac{1}{2n}[, \\ 0 & \text{if} \quad x \in C_{\mathbb{R}}\left(]-\frac{1}{2n}, \frac{1}{2n}[\right). \end{cases}$$

1) Study simple convergence.

2) Show that $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R})$ to the Dirac delta function δ.

Exercice 14.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined by

$$f_n = \frac{\ln x}{1 + x^2 + \dfrac{1}{n}}, \quad x \in I =]0, +\infty[.$$

1− Show that f_n defines a distribution on I.

2− Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(I)$.

Exercice 15.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined for all $n \in \mathbb{N}$ by $f_n(x) = \sqrt{n}e^{-nx^2}$.

Study the convergence of $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}'(\mathbb{R})$. (hint: $\int\limits_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$.)

Exercice 16.

Show that the sequence of distributions $(T_n)_{n\geqslant 1}$ defined by

$$T_n = n\left(\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}\right), \quad \forall n \in \mathbb{N}^\star,$$

converges to $-2\delta'$ in $\mathcal{D}'(\mathbb{R})$.

Exercice 17.

Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} \frac{\sin(nx)}{x} & \text{if} \quad x \neq 0, \\ n & \text{if} \quad x = 0. \end{cases}$$

Show that $(f_n)_{n\in\mathbb{N}}$ converges in the sense of distributions to $\pi\delta$.

Exercice 18.

What are the limits in $\mathcal{D}'(\mathbb{R})$, as $n \to \infty$, of the following sequences of functions

$$f_{\mathrm{n}}(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \quad \text{and} \quad \mathrm{F}_{\mathrm{n}}(x) = \int_{-\infty}^{x} f_n(t)\,\mathrm{d}t.$$

Exercice 19.

For all $n \in \mathbb{N}^*$, we consider the distribution T_n defined by

$$T_n = \delta - \delta_{\frac{1}{n}} - \frac{1}{n}\delta'.$$

1) Show that the sequence of distributions T_n tends to zero in $\mathcal{D}'(\mathbb{R})$.

2) Show that the numerical series with terms $\langle T_n, \varphi \rangle$ converges. Deduce that the series of distributions with terms T_n converges in $\mathcal{D}'(\mathbb{R})$.

Exercice 20.

Study the convergence in $\mathcal{D}'(\mathbb{R})$ of the following series:

$$\sum_{n=0}^{N} \delta_n^{(n)}, \qquad \sum_{n=0}^{N} \delta^{(n)}.$$

Hint: For the second series, take ψ equal to 1 in the neighborhood of 0 and then consider $\varphi(x) = \psi(x)e^x$.

Exercice 21.

Show that

$$\frac{\tau_{-h}T - T}{h} \underset{h \to 0}{\longrightarrow} T',$$

in the sense of distributions, for all $T \in \mathcal{D}'(\mathbb{R})$.

3.5 Correction of the Exercises in Chapter 3

Correction of Exercise 13

1) *Let $x \in \mathbb{R}^*$. There exists $N \in \mathbb{N}$ such that*

$$\forall n > N, \quad \frac{1}{2n} < \frac{1}{2N} \leqslant |x|,$$

which implies that for all $n > N$, $f_n(x) = 0$. In other words, for a certain rank, $f_n(x) = 0$, and thus

$$\lim_{n \to +\infty} f_n(x) = 0 \quad \forall x \in \mathbb{R}^*.$$

2) *<u>Method 1:</u> We have*

$$\int_{-\infty}^{+\infty} f_n(x)\,\mathrm{d}x = \int_{\frac{1}{2n}}^{\frac{1}{2n}} n\,\mathrm{d}x = 1.$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Since $\langle \delta, \varphi \rangle = \varphi(0)$, we have

$$\begin{aligned} |\langle f_n, \varphi \rangle - \langle \delta, \varphi \rangle| &= \left| \int_{-\infty}^{+\infty} \left(f_n(x)\varphi(x) - \varphi(0) \right) \mathrm{d}x \right| \\ &= \left| \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n \left(\varphi(x) - \varphi(0) \right) \mathrm{d}x \right| \\ &\leqslant \sup_{x \in [-\frac{1}{2n}, \frac{1}{2n}]} |\varphi(x) - \varphi(0)| \int_{\frac{1}{2n}}^{\frac{1}{2n}} n\,\mathrm{d}x \\ &\leqslant |\varphi(x_n) - \varphi(0)| \overset{n \to +\infty}{\longrightarrow} 0, \end{aligned}$$

because for all $n \in \mathbb{N}$, there exists $x_n \in [-\frac{1}{2n}, \frac{1}{2n}]$ such that

$$\sup_{x \in [-\frac{1}{2n}, \frac{1}{2n}]} |\varphi(x) - \varphi(0)| = |\varphi(x_n) - \varphi(0)|.$$

<u>Method 2:</u> Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$< f_n, \varphi > -\varphi(0) = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n(\varphi(x) - \varphi(0))\,\mathrm{d}x,$$

Let $y = nx$, then

$$< f_n, \varphi > -\varphi(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\varphi\left(\frac{y}{n}\right) - \varphi(0)\right) \mathrm{d}y,$$

Using the mean value theorem, we deduce that

$$|< f_n, \varphi > -\varphi(0)| \leqslant \frac{1}{2n} \sup_{y\in\mathbb{R}} \left|\varphi\left(\frac{y}{n}\right) - \varphi(0)\right| \leqslant \frac{1}{2n} \sup_{t\in\mathbb{R}} |\varphi'(t)| \underset{n\to+\infty}{\longrightarrow} 0.$$

Correction of Exercise 14

1) *f_n is continuous on $I =]0, +\infty[$, so it belongs to $L^1_{\mathrm{loc}}(I)$, therefore, $f_n \in \mathcal{D}'(I)$.*

2) *As*

$$f_n \longrightarrow f, \quad a.e \text{ in } I,$$

where $f(x) = \dfrac{\ln x}{1 + x^2}$, and there exists $C > 0$ such that

$$|f_n(x)\varphi(x)| \leqslant C|\varphi(x)|, \quad \forall x \in I,$$

where f is in $L^1_{\mathrm{loc}}(I)$, by the dominated convergence theorem, we deduce that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{D}'(I)$.

Correction of Exercise 15

f_n is continuous on $\mathbb{R}$, so $f_n \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $f_n \in \mathcal{D}'(\mathbb{R})$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, using the change of variable $y = \sqrt{n}x$, we obtain

$$\begin{aligned} < f_n, \varphi > &= \sqrt{n} \int_{-\infty}^{+\infty} e^{-nx^2} \varphi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} e^{-y^2} \varphi\left(\frac{y}{\sqrt{n}}\right) \mathrm{d}y. \end{aligned}$$

Applying the dominated convergence theorem to the sequence $e^{-y^2}\varphi\left(\frac{y}{\sqrt{n}}\right)$, which converges almost everywhere in $\mathbb{R}$ to $e^{-y^2}\varphi(0)$ and is dominated by the function $\sup|\varphi(x)|e^{-y^2}$, which belongs to the $L^1(\mathbb{R})$ space, we get

$$< f_n, \varphi > \underset{n\to\infty}{\longrightarrow} \sqrt{\pi}\varphi(0) =< \sqrt{\pi}\delta, \varphi >$$

Correction of Exercise 16

For all $\varphi \in \mathcal{D}(\Omega)$, *we have*

$$\begin{aligned}\left\langle \delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}, \varphi \right\rangle &= \left\langle \delta_{\frac{1}{n}}, \varphi \right\rangle - \left\langle \delta_{-\frac{1}{n}}, \varphi \right\rangle \\ &= \varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right),\end{aligned}$$

Using a Taylor expansion to order 2 *in the neighborhood of* 0, *we obtain*

$$\begin{aligned}\left\langle \delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}, \varphi \right\rangle =&\varphi(0) + \frac{1}{n}\varphi'(0) + \frac{1}{2n^2}\varphi''(0) + o\left(\frac{1}{n^2}\right) \\ &- \left(\varphi(0) - \frac{1}{n}\varphi'(0) + \frac{1}{2n^2}\varphi''(0) - o'\left(\frac{1}{n^2}\right)\right), \\ =&\frac{2}{n}\varphi'(0) + o\left(\frac{1}{n^2}\right) + o'\left(\frac{1}{n^2}\right),\end{aligned}$$

thus

$$\lim_{n\to+\infty} \left\langle n\left(\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}\right), \varphi \right\rangle = 2\varphi'(0) = \langle -2\delta', \varphi\rangle,$$

in $\mathcal{D}'(\Omega)$, *so,*

$$\lim_{n\to+\infty} n\left(\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}\right) = -2\delta'.$$

Correction of Exercise 17

Let $\varphi \in \mathcal{D}(\mathbb{R})$, *then there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$, *then*

$$\langle f_n, \varphi\rangle = \int_{-a}^{a} \frac{\sin(nx)}{x}\varphi(x)\,\mathrm{d}x.$$

Let χ *be the characteristic function of the interval* $[-a, a]$, *then*

$$\langle f_n, \varphi\rangle = \varphi(0)\int_{-na}^{na} \frac{\sin y}{y}\,\mathrm{d}y + \int_{-\infty}^{+\infty} \chi(x)\sin(nx)\frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x$$

As n *tends to infinity, the first integral on the right-hand side converges to* $\pi\delta$, *and the second integral converges to* 0 *by the dominated convergence theorem applied to the sequence* $\chi(x)\sin(nx)\frac{\varphi(x)-\varphi(0)}{x}$, *which converges almost everywhere in* $\mathbb{R}$ *to* 0 *and is bounded by the function* $x \mapsto \chi(x)\frac{\varphi(x)-\varphi(0)}{x}$, *which is integrable on* $\mathbb{R}$.

Correction of Exercise 18

f_n is continuous on $\mathbb{R}$, then $f_n \in L^1_{\text{loc}}(\Omega)$, thus $f_n \in \mathcal{D}'(\Omega)$. Let $\varphi \in \mathcal{D}(\Omega)$, using the change of variable $u = nx$, we obtain

$$\begin{aligned}\langle f_n, \varphi \rangle &= \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-n^2x^2}\varphi(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2}\varphi\left(\frac{u}{n}\right)\,\mathrm{d}u.\end{aligned}$$

By the dominated convergence theorem applied to the sequence $e^{-u^2}\varphi\left(\frac{u}{\sqrt{n}}\right)$, which converges almost everywhere in $\mathbb{R}$ to $e^{-u^2}\varphi(0)$ and is dominated by the function $\sup|\varphi(x)|e^{-u^2}$ that belongs to the $L^1(\mathbb{R})$ space, we have

$$\langle f_n, \varphi \rangle \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2}\varphi(0)\,\mathrm{d}u = \varphi(0) = <\delta, \varphi>,$$

$(f_n)_{n\in\mathbb{N}}$ converges to the Dirac delta function in $\mathcal{D}'(\mathbb{R})$.

We have

$$\mathrm{F_n}(x) = \int_{-\infty}^{x} f_n(t)\,\mathrm{d}t = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{nx} e^{-u^2}\,\mathrm{d}u,$$

the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function F defined by

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \frac{1}{2} & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

By the dominated convergence theorem, we deduce that

$$\int_{-\infty}^{+\infty} F_n(x)\varphi(x)\,\mathrm{d}x \to \int_{0}^{+\infty} \varphi(x)\,\mathrm{d}x = \int_{0}^{+\infty} H(x)\varphi(x)\,\mathrm{d}x$$

$(F_n)_{n\in\mathbb{N}}$ converges to the function H in $\mathcal{D}'(\mathbb{R})$.

Correction of Exercise 19

1) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$, *we have*

$$
\begin{aligned}
> \langle T_n, \varphi \rangle &= \left\langle \delta - \delta_{\frac{1}{n}} - \frac{1}{n}\delta', \varphi \right\rangle \\
&= \langle \delta, \varphi \rangle - \left\langle \delta_{\frac{1}{n}}, \varphi \right\rangle - \frac{1}{n} \langle \delta', \varphi \rangle \\
&= \langle \delta, \varphi \rangle - \left\langle \delta_{\frac{1}{n}}, \varphi \right\rangle + \frac{1}{n} \langle \delta, \varphi' \rangle \\
&= \varphi(0) - \varphi\left(\frac{1}{n}\right) + \frac{1}{n}\varphi'(0),
\end{aligned}
$$

Since φ *is continuous, we have*

$$\lim_{n\to+\infty} \varphi\left(\frac{1}{n}\right) = \varphi(0)$$

Therefore,

$$\lim_{n\to+\infty} \langle T_n, \varphi \rangle = 0,$$

the sequence of distributions T_n *tends to* 0 *in* $\mathcal{D}'(\mathbb{R})$.

2) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$ *and* $n \in \mathbb{N}^*$, *applying the Taylor series to the function* φ *on the interval* $[0, \frac{1}{n}]$ *to order two gives*

$$\varphi\left(\frac{1}{n}\right) = \varphi(0) + \frac{1}{n}\varphi'(0) + \frac{1}{2n^2}\varphi''(\xi_n),$$

where $\xi_n \in [0, \frac{1}{n}]$. *then*

$$\langle T_n, \varphi \rangle = -\frac{1}{2n^2}\varphi''(\xi_n), \quad \forall n \in \mathbb{N}^*,$$

as φ *is of class* $\mathcal{C}^2$ *and has compact support on* $\mathbb{R}$, *we can set*

$$M = \frac{1}{2}\sup_{x\in\mathbb{R}} |\varphi''(x)|.$$

so,

$$\forall n \in \mathbb{N}^*, \quad |\langle T_n, \varphi \rangle| \leqslant \frac{M}{n^2}.$$

The series with terms $\frac{1}{n^2}$ *is convergent, by the comparison test, the numerical series with terms* $\langle T_n, \varphi \rangle$ *is absolutely convergent, and thus convergent.*

Since, for all φ, *the series with terms* $\langle T_n, \varphi \rangle$ *is convergent, we deduce that the series of distributions with terms* T_n *converges in* $\mathcal{D}'(\mathbb{R})$.

Correction of Exercise 20

1) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, we have*

$$\left\langle \sum_{n=0}^{N} \delta_n^{(n)}, \varphi \right\rangle = \sum_{n=0}^{N} (-1)^n \varphi^{(n)}(n) \underset{N\to+\infty}{\longrightarrow} \sum_{n=0}^{+\infty} (-1)^n \varphi^{(n)}(n)$$

where the term on the right is well-defined; it is an example of a distribution of infinite order (see [reference 21, 6]).

2) *On the other hand, the second series does not converge in* $\mathcal{D}'(\mathbb{R})$*. To see this, it is sufficient to consider* $\psi \in \mathcal{D}(\mathbb{R})$ *equal to 1 in a neighborhood of 0 and evaluate the series in question with the test function* $\varphi(x) = \psi(x)e^x$*. Indeed, for all* $n \in \mathbb{N}$*, by the Leibniz formula,* $\varphi^{(n)}(0) = 1$*, which ensures that the series*

$$\left\langle \sum_{n=0}^{N} \delta^{(n)}, \varphi \right\rangle = \sum_{n=0}^{N} (-1)^n \varphi^{(n)}(0) = \sum_{n=0}^{N} (-1)^n$$

is divergent.

Correction of Exercise 21

We have the following convergence

$$\frac{\tau_{-h}T - T}{h} \underset{h\to 0}{\longrightarrow} T', \quad in$$

in $\mathcal{D}'(\mathbb{R})$*, if and only if, for all* $\varphi \in \mathcal{D}(\mathbb{R})$

$$\left\langle \frac{\tau_{-h}T - T}{h}, \varphi \right\rangle \underset{h\to 0}{\longrightarrow} \langle T', \varphi \rangle\,;$$

We have

$$\begin{aligned}
\left\langle \frac{\tau_{-h}T - T}{h}, \varphi \right\rangle &= \left\langle \frac{\tau_{-h}T}{h}, \varphi \right\rangle - \left\langle \frac{T}{h}, \varphi \right\rangle \\
&= \left\langle \tau_{-h}T, \frac{\varphi}{h} \right\rangle - \left\langle T, \frac{\varphi}{h} \right\rangle \\
&= \left\langle T, \frac{\tau_h\varphi}{h} \right\rangle - \left\langle T, \frac{\varphi}{h} \right\rangle \\
&= \left\langle T, \frac{\tau_h\varphi - \varphi}{h} \right\rangle
\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\tau_h\varphi(x)-\varphi(x)}{h} &= \frac{\varphi(x-h)-\varphi(x)}{h} \\ &= -\frac{\varphi(x+h)-\varphi(x)}{h},\end{aligned}$$

then

$$-\frac{\varphi(x+h)-\varphi(x)}{h} \underset{h\to 0}{\longrightarrow} -\varphi'$$

in $\mathcal{D}(\mathbb{R})$. Due to the continuity of T, we deduce that

$$\left\langle T, \frac{\tau_h\varphi-\varphi}{h}\right\rangle \underset{h\to 0}{\longrightarrow} -\langle T, \varphi'\rangle .$$

CHAPTER 4

OPERATIONS ON DISTRIBUTIONS

4.1 Introduction

In this Chapter, we will delve into operations on distributions. Several operations defined for functions can be extended in a similar manner for distributions. We also introduce the concept of multiplication of a distribution by a C^∞ function, which is a crucial ingredient when dealing with differential equations with variable coefficients.

Similar to the concept of a function's support, we define the support of a distribution. This is a somewhat technical aspect, but it is important as it has applications in various mathematical branches like differential geometry and Lie group theory.

Chapter Objectives

- *Understand how to solve differential equations in $\mathcal{D}'$.*
- *Get acquainted with the notion of the support of a distribution.*

4.2 Operations on Distributions

In all that follows, let $T \in \mathcal{D}'(\Omega)$.

4.1 Definition (Addition)

*The **sum** of two distributions T_1 and T_2 in $\mathcal{D}'(\Omega)$ is defined as:*

$$\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

4.2 Definition (Multiplication by a Constant)

*The **multiplication** of a distribution by a constant $\lambda \in \mathbb{C}$ is defined as:*

$$\langle \lambda T, \varphi \rangle = \langle T, \lambda \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 4.1. *Based on the definitions above, we equip the space $\mathcal{D}'(\Omega)$ with the structure of a vector space over $\mathbb{C}$.*

4.3 Definition (Restriction to an Open Set)

*Let U be an open set in Ω. The **restriction** of T to U, denoted as $T_{|U}$, is a distribution on U defined by:*

$$\langle T_{|U}, \varphi \rangle = \langle T, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(U).$$

This definition makes sense because every element φ in $\mathcal{D}(U)$ can be considered as an element of $\mathcal{D}(\Omega)$ by extending it as zero outside of U. The continuity can be easily verified.

Example 4.1. $\delta_{|\mathbb{R}^\star} = 0$. *In fact, for all $\varphi \in \mathcal{D}(\mathbb{R}^\star)$, with $\operatorname{supp} \varphi \subset \mathbb{R}^\star$, we have $\varphi(0) = 0$, and thus:*

$$\langle \delta_{|\mathbb{R}^\star}, \varphi \rangle = \varphi(0) = 0.$$

Exercise 4.1. *Show that* $\operatorname{vp}\left(\frac{1}{x}\right)_{|\mathbb{R}_+^\star} = \frac{1}{x}$.

4.4 Definition (Translation of a Function)

*Let $f : \mathbb{R}^N \to \mathbb{C}$ be a function, and $h \in \mathbb{R}^N$. The **translation** $\tau_h f$ of f by h is the function defined as:*

$$\tau_h f(x) = f(x - h), \quad \forall x \in \mathbb{R}^N.$$

Assuming that $f \in L^1_{loc}(\mathbb{R})$, for all $\varphi \in \mathcal{D}(\mathbb{R})$, the associated regular distribution of $\tau_h f$ is given by:

$$\begin{aligned}\langle T_{\tau_h f}, \varphi\rangle &= \int_{\mathbb{R}} f(x-h)\varphi(x)\,dx \\ &= \int_{\mathbb{R}} f(x)\varphi(x+h)\,dx \\ &= \int_{\mathbb{R}} f(x)\tau_{-h}\varphi(x)\,dx \\ &= \langle T_f, \tau_{-h}\varphi\rangle .\end{aligned}$$

This justifies the following definition.

4.5 Definition (Translation of a Distribution/Periodic Distribution)

We call the **translation** of T, denoted as $\tau_h T$, the distribution defined as:

$$\langle \tau_h T, \varphi\rangle = \langle T, \tau_{-h}\varphi\rangle , \quad \forall \varphi \in \mathcal{D}(\Omega).$$

The distribution T is called **periodic** with period h if $\tau_h T = T$.

Example 4.2. *For all $\varphi \in \mathcal{D}(\Omega)$, we have*

$$\begin{aligned}\langle \tau_h\delta, \varphi\rangle &= \langle \delta, \tau_{-h}\varphi\rangle \\ &= \tau_{-h}\varphi(0) \\ &= \varphi(h) \\ &= \langle \delta_h, \varphi\rangle ,\end{aligned}$$

then

$$\tau_h\delta \overset{\mathcal{D}'}{=} \delta_h.$$

4.6 Definition (Symmetric of a Function)

Let the function $f : \mathbb{R}^N \to \mathbb{C}$. The **symmetric** $\hat{f}$ of f is the function defined as:

$$\hat{f}(x) = f(-x), \quad \forall x \in \mathbb{R}^N.$$

The graph of $\hat{f}$ is symmetric to that of f with respect to the yy' axis.

Let $f \in L^1_{loc}(\mathbb{R})$. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have:

$$\begin{aligned}\left\langle \hat{f}, \varphi \right\rangle &= \int_{-\infty}^{+\infty} f(-x)\varphi(x)\,dx \\ &= \int_{-\infty}^{+\infty} f(x)\varphi(-x)\,dx \\ &= \langle f, \hat{\varphi} \rangle .\end{aligned}$$

4.7 Definition (Symmetric of a Distribution/Parity)

We call the **symmetric** of T, denoted as $\hat{T}$, the distribution defined as:

$$\left\langle \hat{T}, \varphi \right\rangle = \langle T, \hat{\varphi} \rangle , \quad \forall \varphi \in \mathcal{D}(\Omega).$$

The distribution T is called **even** if $\hat{T} = T$, and it is called **odd** if $\hat{T} = -T$.

Example 4.3.

For all $\varphi \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned}\left\langle \hat{\delta}, \varphi \right\rangle &= \langle \delta, \hat{\varphi} \rangle \\ &= \varphi(0) \\ &= \langle \delta, \varphi \rangle .\end{aligned}$$

So, $\hat{\delta} = \delta$, and therefore, δ is an even distribution.

The distribution δ' is odd, and it can be easily shown that $\hat{\delta}' = -\delta'$.

Remark 4.2. *Every distribution can be expressed as the sum of an even distribution and an odd distribution.*

There is no multiplication possible between two arbitrary distributions. However, the multiplication of a distribution with a C^∞ function can be defined.

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Let $g \in C^\infty(\Omega)$ and $f \in L^1_{loc}(\Omega)$. Then $gf \in L^1_{loc}(\Omega)$, and for all $\varphi \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned}\langle gf, \varphi \rangle &= \int_\Omega f(x)g(x)\varphi(x)\,dx \\ &= \langle f, g\varphi \rangle.\end{aligned}$$

We can extend this property to distributions.

4.8 Definition (Multiplication by a C^∞ Function)

Let $T \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. We define the **product fT of a distribution and a function** as follows:

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 4.3.

$\langle fT, \varphi \rangle$ is well-defined because $f\varphi \in \mathcal{D}(\Omega)$ for $\varphi \in \mathcal{D}(\Omega)$ and $f \in C^\infty(\Omega)$.

Furthermore, if $\varphi \in \mathcal{D}_K(\Omega)$ for a compact set K in Ω, then $f\varphi \in \mathcal{D}_K(\Omega)$, and we have:

$$|\langle fT, \varphi \rangle| = |\langle T, f\varphi \rangle| \leqslant C \sup_{x \in K, |\alpha| \leqslant m} |D^\alpha (f\varphi)|.$$

Using the Leibniz formula, we obtain:

$$\sup_{x \in K, |\alpha| \leqslant m} |D^\alpha (f\varphi)| \leqslant C \sup_{x \in K, |\alpha| \leqslant m} |D^\alpha f| \sup_{x \in K, |\alpha| \leqslant m} |D^\alpha \varphi|,$$

which means:

$$|\langle fT, \varphi \rangle| \leqslant C' \sup_{x \in K, |\alpha| \leqslant m} |D^\alpha \varphi|, \quad \forall \varphi \in \mathcal{D}_K(\Omega),$$

where $C' = C \sup_{x \in K, |\alpha| \leqslant m} |D^\alpha f|$. This implies that $fT \in \mathcal{D}'(\Omega)$ and the order of fT is lower than the order of T.

Example 4.4.

1) *Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function that never vanishes. For all distributions T, we have:*

$$fT = 0 \Longleftrightarrow T = 0.$$

2) *Let* $f \in C^\infty(\Omega)$ *and* $a \in \Omega$. *For all* $\varphi \in \mathcal{D}(\Omega)$, *we have:*

$$\begin{aligned}\langle f\delta_a, \varphi\rangle &= \langle \delta_a, f\varphi\rangle \\ &= \langle f(a)\delta_a, \varphi\rangle .\end{aligned}$$

Then,

$$f\delta_a = f(a)\delta_a.$$

In particular,

$$x\delta = 0.$$

3) *For all* $x \in \mathbb{R}$, *we have:*

$$x\,\mathrm{vp}\,\frac{1}{x} = 1.$$

Recall that the principal value of Cauchy, $\mathrm{vp}\,\dfrac{1}{x}$, *is the distribution defined by:*

$$\forall\varphi \in \mathcal{D}(\mathbb{R}), \quad \left\langle \mathrm{vp}\,\frac{1}{x}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\int\limits_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx.$$

So, for all $\varphi \in \mathcal{D}(\mathbb{R})$, *we have:*

$$\begin{aligned}\left\langle x\,\mathrm{vp}\,\frac{1}{x}, \varphi\right\rangle &= \left\langle \mathrm{vp}\,\frac{1}{x}, x\varphi\right\rangle \\ &= \lim_{\varepsilon\to 0}\int\limits_{|x|>\varepsilon} \varphi(x)\,dx \\ &= \lim_{\varepsilon\to 0}\left\{\int\limits_{-\infty}^{-\varepsilon}\varphi(x)\,dx + \int\limits_{\varepsilon}^{+\infty}\varphi(x)\,dx\right\} \\ &= \int\limits_{-\infty}^{+\infty} 1\varphi(x)\,dx \\ &= \langle 1, \varphi\rangle ,\end{aligned}$$

where 1 *denotes the regular distribution associated with the constant function* $x \mapsto 1$.

Remark 4.4. *The relation* $x\,\mathrm{vp}\,\frac{1}{x} = 1$ *is extremely interesting because it allows us to deduce that the inverse of* x *in the sense of distributions is the distribution* $\mathrm{vp}\,\frac{1}{x}$ *and not* $\frac{1}{x}$ *(which defines a regular distribution).*

Let $f \in C(\mathbb{R})$ *and* $T \in \mathcal{D}'(\mathbb{R})$*. By applying integration by parts, we obtain:*

$$(fT)' = fT' + f'T,$$

in the sense of distributions.

4.1 Proposition (Equation $xT = 0$*)*

Let $T \in \mathcal{D}'(\mathbb{R})$*. If* $xT = 0$*, then* $T = c\delta$*, where* $c \in \mathbb{R}$*.*

Proof.

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi = 1$ in the neighborhood of 0. Define the function ψ as follows:

$$\psi(x) = \begin{cases} \frac{\varphi(x) - \varphi(0)\chi(x)}{x} & \text{if } x \neq 0, \\ \varphi'(0) & \text{if } x = 0. \end{cases}$$

Then $\psi \in \mathcal{D}(\mathbb{R})$, and we have:

$$\varphi(x) = \varphi(0)\chi(x) + x\psi(x),$$

As $xT = 0$, we get:

$$\langle T, \varphi \rangle = \varphi(0) \langle T, \chi \rangle,$$

Therefore,

$$T = c\delta,$$

where $c = \langle T, \chi \rangle$. □

Example 4.5.

The function $x \to \frac{1}{x^2}$ *is not locally integrable on* $\mathbb{R}$*, and therefore, it cannot be* *associated with a regular distribution. To eliminate the divergent part of the integral* $\int_0^{+\infty} \frac{\varphi(x)}{x^2}$ *for* $\varphi \in \mathcal{D}(\mathbb{R})$*, we define the finite part as:*

$$\left(\mathrm{Pf}\,\frac{1}{x^2}, \varphi\right\rangle = \int_0^{+\infty} \frac{\varphi(x)+\varphi(-x)-2\varphi(0)}{x^2}\,dx.$$

It can be easily verified that we have:

$$x\,\mathrm{Pf}\,\frac{1}{x^2} = \mathrm{vp}\,\frac{1}{x} \quad \text{and } x^2\,\mathrm{Pf}\,\frac{1}{x^2} = 1.$$

4.2 Proposition (Leibniz's formula)

Let $T \in \mathcal{D}'(\Omega)$*,* $f \in C^\infty(\Omega)$*, and* $i \in \{1, \ldots, N\}$*, then*

$$\frac{\partial}{\partial x_i}(fT) = \frac{\partial f}{\partial x_i}T + f\frac{\partial T}{\partial x_i}.$$

Proof.

For all $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_i}(fT), \varphi \right\rangle &= -\left\langle fT, \frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= -\left\langle T, f\frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= -\left\langle T, \frac{\partial}{\partial x_i}(f\varphi) - \frac{\partial f}{\partial x_i}, \varphi \right\rangle \\
&= \left\langle \frac{\partial T}{\partial x_i}, f\varphi \right\rangle + \left\langle \frac{\partial f}{\partial x_i}T, \varphi \right\rangle \\
&= \left\langle \frac{\partial f}{\partial x_i}T + f\frac{\partial T}{\partial x_i}, \varphi \right\rangle.
\end{aligned}$$

□

Generalization (Leibniz's formula)

Let $T \in \mathcal{D}'(\Omega)$*, and*

$$D^\alpha(f\varphi) = \sum_{\beta \leqslant \alpha} C_\alpha^\beta D^\beta f D^{\alpha-\beta}\varphi.$$

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4.3 Distributions with Compact Support

In this section, we will extend the bracket $\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}}$ to the case where the function φ does not have compact support, which will allow us to define the convolution of distributions.

4.9 Definition (Support of a distribution/nullification open set)

*The **support of a distribution** T is denoted by $\operatorname{supp} T = C_\Omega w$, where w is the largest open set on which T is zero.*

*w is called the **nullification open set** of T.*

The existence of the open set w is guaranteed by the following proposition:

4.3 Proposition

Let $T \in \mathcal{D}'(\Omega)$. There exists a largest open set $w \subset \Omega$ such that the restriction $T_{|w}$ is zero.

Proof.

Let $(w_i)_{i\in I}$ be a family of open sets in Ω such that $T_{|w_i} = 0$. Let w be their union. We need to show that $T_{|w} = 0$. For this, let $\varphi \in \mathcal{D}(w)$. Since $\operatorname{supp}(\varphi)$ is a compact set included in U, there exists a finite number of open sets $w_1, \ldots, w_n$ such that

$$\operatorname{supp}(\varphi) \subset \bigcup_{i=1}^{n} w_i \text{ and } T_{|w_i} = 0, \quad \forall i = 1, \ldots, n.$$

Let $(\rho_i)_{i=1,\ldots,n}$ be a partition of unity associated with the covering of $\operatorname{supp}(\varphi)$ by the open sets $(w_i)_{1\leqslant i\leqslant n}$, then $\varphi = \sum_{i=1}^{n} \rho_i\varphi$ with $\rho_i\varphi \in \mathcal{D}(w_i)$. Consequently,

$$\langle T, \varphi\rangle = \sum_{i=1}^{n} \langle T, \rho_i\varphi\rangle = \sum_{i=1}^{n} \left\langle T_{|w_i}, \rho_i\varphi\right\rangle = 0.$$

□

Example 4.6.

The support of a regular distribution T_f is identical to the support of the function $f \in L^1_{loc}(\Omega)$, meaning

$$\operatorname{supp}(T_f) = \operatorname{supp}(f).$$

Indeed, if w *is an open set of* Ω *on which* T *vanishes, then:*

$$\langle T, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{D}(w).$$

if and only if:

$$\int_\Omega f(x)\varphi(x)\, dx = 0,$$

which is equivalent to:

$$f(x) = 0 \text{ a.e. in } w.$$

Example 4.7.

The support of the distribution associated with the Heaviside function $[0, +\infty[$

Example 4.8.

The support of δ_a *is* $\{a\}$*, called a point support. Indeed, for all* $\varphi \in \mathcal{D}(\Omega \setminus \{a\})$*, we have:*

$$\langle \delta_a, \varphi \rangle = \varphi(a) = 0,$$

thus $\operatorname{supp} \delta_a \subset \{a\}$*. Conversely, there exists* $\psi = 1$ *in the neighborhood of* a*, then:*

$$\langle \delta_a, \psi \rangle = \psi(a) = 1,$$

so: $a \in \operatorname{supp} \delta_a$.

Example 4.9.

Let $\alpha \in \mathbb{N}^n$*,* $a \in \Omega$*, and* T *is the distribution defined by*

$$\langle T, \varphi \rangle = D^\alpha \varphi(a), \quad \forall \varphi \in \mathcal{D}(\Omega).$$

then $\operatorname{supp}(T) = \{a\}$.

Indeed, if $\varphi \in \mathcal{D}(\Omega \setminus \{a\})$*, we have*

$$\langle T, \varphi \rangle = 0,$$

so $\operatorname{supp}(T) \subset \{a\}$*. To prove that* $a \in \operatorname{supp}(T)$*, consider an open neighborhood* V *of* a *and* $\chi \in \mathcal{D}(V)$ *such that* $\chi = 1$ *in the neighborhood of* a*. Let*

$$\varphi(x) = \frac{(x-a)^\alpha}{\alpha!}\chi(x),$$

then $\varphi \in \mathcal{D}(V)$. Using the Leibniz formula, we find that $D^\alpha\varphi(a) = 1$; therefore, $a \in \operatorname{supp}(T)$.

Let $\mathcal{E}'(\Omega)$ be the topological dual of the functional space $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ equipped with its natural topology.

4.1 Theorem

The injective application

$$\mathrm{Id} : \mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$$

is continuous.

Proof.

Consider (K_i) as an exhaustive family of compact subsets of Ω. By definition, the injection $\mathcal{D}_{K_i}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ is continuous. According to Proposition 2.5.1, it follows that Id is continuous since $\mathcal{D}(\Omega) = \bigcup_i \mathcal{D}_{K_i}(\Omega)$.

As a consequence of this result, every bounded subset of $\mathcal{D}(\Omega)$ is a bounded subset of the space $\mathcal{E}(\Omega)$. Furthermore, □

4.2 Theorem

The set $\mathcal{D}(\Omega)$ is a dense subspace in $\mathcal{E}(\Omega)$.

Proof.

Let (K_i) be an exhaustive family of compact subsets of Ω. There exists a family of functions in $\mathcal{D}(\Omega)$, denoted as (β_i), such that $\beta_i \equiv 1$ in each neighborhood of K_i. If $\varphi \in \mathcal{D}(\Omega)$, let $\varphi_i = \beta_i\varphi \in \mathcal{D}(\Omega)$. It is easy to verify that $\varphi_i \to \varphi$ in $\mathcal{D}(\Omega)$.

On the other hand, we have the injection

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$$

So, any element $T \in \mathcal{E}'(\Omega)$ defines a distribution on Ω. □

4.3 Theorem

$T \in \mathcal{E}'(\Omega)$ if and only if there exists $C > 0$, an integer $m \geqslant 0$, and a compact set $K \subset \Omega$ such that:

$$|\langle T, \varphi\rangle| \leqslant C \sup_{|\alpha| \leqslant m, x \in K} |D^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{E}(\Omega).$$

Proof.

If $T \in \mathcal{E}'(\Omega)$, there exists a neighborhood of 0 in $\mathcal{E}(\Omega)$ of the form

$$V = \{\varphi \in \mathcal{E}(\Omega), P_{K,m}(\varphi) \leqslant \varepsilon\},$$

such that

$$|\langle T, \varphi\rangle| \leqslant 1, \quad \forall \varphi \in V.$$

Choose φ such that $P_{K,m}(\varphi) \neq 0$. Then, $\varepsilon / P_{K,m}(\varphi) \in V$. It follows that

$$|\langle T, \varphi\rangle| \leqslant \varepsilon^{-1} P_{K,m}(\varphi)$$

On the other hand, if $\varphi \in \mathcal{E}(\Omega)$ satisfies $P_{K,m}(\varphi) = 0$, then $\langle T, \varphi\rangle = 0$. Indeed, such a function φ is in V as well as the functions $\lambda\varphi, \lambda \in \mathbb{K}$. If $\langle T, \varphi\rangle \neq 0$, then $|\langle T, \lambda\varphi\rangle|$ can be chosen arbitrarily large, contradicting the first inequality. Therefore, the first inequality holds for all $\varphi \in \mathcal{E}(\Omega)$. The rest of the proof is straightforward. □

More precisely, we will show that:

4.4 Theorem

The elements of the functional space $\mathcal{E}'(\Omega)$ are distributions with compact supports contained in Ω.

Proof.

In the previous proof, we showed that if $T \in \mathcal{E}'(\Omega)$, there exists $\varepsilon \geqslant 0$, an integer $m \geqslant 0$, and a compact set K in Ω such that for all $\varphi \in \mathcal{E}(\Omega)$ satisfying $P_{K,m}(\varphi) \leqslant \varepsilon$, then $|\langle T, \varphi\rangle| \leqslant 1$. We also noticed that for all $\varphi \in \mathcal{E}(\Omega)$ satisfying $P_{K,m}(\varphi) = 0$, we have $|\langle T, \varphi \rangle| = 0$. Since all $\varphi \in \mathcal{D}(\Omega \setminus K)$ satisfy this condition, it follows that T vanishes on $\Omega \setminus K$, and thus the support of T is contained in K. $\square$

4.4 Exercises on Chapter 4

Exercice 22.

Let I be an open set in $\mathbb{R}$. Show that

1) $T_n \to T$ in $\mathcal{D}'(I)$, and $f \in C^\infty(I)$. Then $fT_n \to fT$ in $\mathcal{D}'(I)$.

2) $T \in \mathcal{D}'(I)$, and $f_n \to f$ in $C^\infty(I)$. Then $f_nT \to fT$ in $\mathcal{D}'(I)$.

3) For all $f \in C^\infty(I)$ and $T \in \mathcal{D}'(I)$, we have

$$\operatorname{supp}(fT) = \operatorname{supp} f \cap \operatorname{supp} T.$$

In particular, if $\operatorname{supp} f \cap \operatorname{supp} T = \varnothing$, then $fT = 0$.

Exercice 23.

Solve the following equations in $\mathcal{D}'(\mathbb{R})$:

1) $xT = 1$.

2) $x^2T = 1$.

Exercice 24.

Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function, and δ be the Dirac measure.

1) Verify that $f\delta = f(0)\delta$, and deduce that $x\delta = 0$.

2) Verify that $f\delta' = f(0)\delta^3 - f'(0)\delta$, and deduce $x\delta' = -\delta$, $x^2\delta' = 0$, and then $x\delta^{(m)} = -\delta^{(m-1)}$.

3) Show that $\frac{d}{\mathrm{d}x}(fT) = \frac{df}{\mathrm{d}x}T + f\frac{dT}{\mathrm{d}x}$.

4) Verify that $x \operatorname{vp} \frac{1}{x} = 1$.

5) Show that, for a distribution T on $\mathbb{R}$ to satisfy $xT = 0$, it is necessary and sufficient for T to be proportional to δ.

6) Find the distributions T such that $xT = 1$.

Exercice 25.

Solve the following differential equations in $\mathcal{D}'(\mathbb{R})$:

1) $T' - cT = \delta.$

2) $T' + xT = \delta.$

3) $T' + T = H.$

4) $T' + e^{-x}T = \delta.$

5) $T' + fT = u$, where T, u are distributions, and $f \in C^{\infty}(\mathbb{R})$.

6) $T'' - c^2 T = \delta.$

Exercice 26.

Provide an example of a distribution $T \in \mathcal{D}'(\mathbb{R})$ and a function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $supp(T) = \{0\}$, $\varphi(0) = 0$, and $\langle T, \varphi \rangle \neq 0$. More generally, if $T \in \mathcal{D}'(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R})$, what conditions are required for T and ψ such that $\langle T, \psi \rangle = 0$? What conditions should be imposed on ψ_1 and $\psi_2 \in \mathcal{D}(\mathbb{R})$ for $\langle T, \psi_1 \rangle = \langle T, \psi_2 \rangle$?

Exercice 27.

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Find the support of the following distributions.

1) $T_1 = \delta.$

2) $\langle T_2, \varphi \rangle = \int_{-1}^{1} x\varphi'(x)\mathrm{d}x.$

3) $\langle T_3, \varphi \rangle = \int_{-\infty}^{+\infty} \cos x \varphi(x)\mathrm{d}x.$

4) $\langle T_5, \varphi \rangle = \int_{-1}^{1} sgn(x)\varphi'(x)\mathrm{d}x.$

Exercice 28.

For all $\varphi \in \mathcal{D}(\mathbb{R})$, consider the following distributions.

1) $\langle T_1, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi\left(x^2\right)\, \mathrm{d}x.$

2) $\langle T_2, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi'(x)\cos x \,\mathrm{d}x.$

3) $\langle T_3, \varphi\rangle = \int\limits_{0}^{\pi} \varphi\left(x^2\right)\, \mathrm{d}x.$

4) $\langle T_4, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi'(x)\ln(x)\,\mathrm{d}x.$

Are these distributions regular? Provide their order and support.

Exercice 29.

Let T be the application from $\mathcal{D}(\mathbb{R}^2)$ to $\mathbb{C}$ defined by

$$\langle T, \varphi\rangle = \int\limits_{0}^{1} \varphi(x, x+1)\,\mathrm{d}x.$$

1) Show that T is a distribution of finite order.

2) Determine the support of T. Is T of compact support?

Exercice 30.

Let T be the linear mapping from $\mathcal{D}(\mathbb{R})$ to $\mathbb{C}$ defined by

$$\langle T, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi(x, -x)\,\mathrm{d}x.$$

1) Show that $T \in \mathcal{D}'(\mathbb{R}^2)$. What is its order?

2) Determine the support of T. Deduce that there are no continuous functions on $\mathbb{R}^2$ whose associated distribution is T.

3) Calculate

$$\frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}.$$

4.5 Correction of the Exercises in Chapter 4

Correction of Exercise 22

1) *For all $\varphi \in \mathcal{D}(I)$, the function fT belongs to $\mathcal{D}(I)$, and we have*

$$\langle fT_n, \varphi\rangle = \langle T_n, f\varphi\rangle = \langle T, f\varphi\rangle = \langle fT, \varphi\rangle \quad \text{as } n \to \infty.$$

2) *Recall that $f_n \to f$ in $C^\infty(I)$ if and only if for all integers $k \in \mathbb{N}$ and all compact subsets $K \subset I$,*

$$\sup_{x\in K}\left|f_n^{(k)}(x) - f^{(k)}(x)\right| \to 0 \quad \text{as } n \to \infty.$$

Since $\varphi \in \mathcal{D}(I)$, then $f_n\varphi \to f\varphi$ in $\mathcal{D}(I)$. Thus, for all $\varphi \in \mathcal{D}(I)$,

$$\langle f_nT, \varphi\rangle = \langle T, f_n\varphi\rangle \to \langle T, f\varphi\rangle = \langle fT, \varphi\rangle \quad \text{as } n \to \infty.$$

3) *Let $O = I \setminus (\operatorname{supp} f \cap \operatorname{supp} T)$ and $\varphi \in C_0^\infty(O)$. As*

$$\operatorname{supp}(f\varphi) = \operatorname{supp} f \cap \operatorname{supp}\varphi \subset I \setminus \operatorname{supp} T,$$

then

$$\langle fT, \varphi\rangle = \langle T, f\varphi\rangle = 0.$$

We conclude that fT is zero on any function $\varphi \in C_0^\infty(O)$, which proves the desired result.

Correction of Exercise 23

1) *Let T_1 and T_2 be two solutions in $\mathcal{D}'(\mathbb{R})$ to the equation $xT = 1$. Then $T_2 - T_1$ is a solution to $xT = 0$, which implies*

$$T_2 = T_1 + c\delta,$$

where $c \in \mathbb{C}$. Furthermore, $T_1 = \operatorname{vp}\left(\frac{1}{x}\right)$ is a solution to the equation $xT = 1$. This leads to the conclusion that the set of solutions is

$$S = \left\{\operatorname{vp}\left(\frac{1}{x}\right) + c\delta, c \in \mathbb{C}\right\}.$$

2) *Similarly, we can express solutions in the form*

$$T_1 + c_1\delta + c_2\delta',$$

where T_1 *is any solution, and* c_1 *and* c_2 *are complex numbers. So, we just need to find* T_1 *such that* $x^2T_1 = 1$*. Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, then there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$*, and we have*

$$\begin{aligned}
\left\langle -x^2 \operatorname{vp}'\left(\frac{1}{x}\right)\varphi \right\rangle &= \left\langle \operatorname{vp}\left(\frac{1}{x}\right), \left(x^2\varphi\right)' \right\rangle \\
&= \left\langle \operatorname{vp}\left(\frac{1}{x}\right), 2x\varphi + x^2\varphi' \right\rangle \\
&= \langle 2, \varphi\rangle + \int_{-a}^{a} x\varphi'(x)\,\mathrm{d}x \\
&= \langle 1, \varphi\rangle.
\end{aligned}$$

Thus, the set of solutions is

$$S = \left\{ \operatorname{vp}'\left(\frac{1}{x}\right) + c_1\delta + c_2\delta', c_1, c_2 \in \mathbb{C} \right\}.$$

Correction of Exercise 24

1) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, we have*

$$\langle f\delta, \varphi\rangle = \langle \delta, f\varphi\rangle = f(0)\varphi(0) = \langle f(0)\delta, \varphi\rangle,$$

thus $f(x)\delta) = f(0)\delta$*, so,* $x\delta = 0$*.*

2) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, and* f *be a* C^1 *function on* $\mathbb{R}$*, then*

$$\begin{aligned}
\langle f(0)\delta' - f'(0)\delta, \varphi\rangle &= -f(0)\varphi'(0) - f'(0)\varphi(0) \\
&= -(f\varphi)'(0) \\
&= \langle \delta', f\varphi\rangle,
\end{aligned}$$

then

$$\langle f(0)\delta' - f'(0)\delta, \varphi\rangle = \langle f\delta', \varphi\rangle.$$

For $f(x) = x$, and then $f(x) = x^2$, we obtain

$$x\delta' = -\delta, x^2\delta' = 0,$$

We can show by induction that

$$x\delta^{(m)} = -\delta^{(m-1)}, \quad \forall m \in \mathbb{N}.$$

3) *Let $\varphi \in \mathcal{D}(\mathbb{R})$, we have*

$$\begin{aligned}
\left\langle \frac{df}{dx}T + f\frac{dT}{dx}, \varphi \right\rangle &= \left\langle T, \frac{df}{dx}\varphi \right\rangle + \left\langle \frac{dT}{dx}, f\varphi \right\rangle \\
&= \left\langle T, \frac{df}{dx}\varphi \right\rangle - \left\langle T, \frac{d}{dx}(f\varphi) \right\rangle \\
&= -\left\langle T, f\varphi' \right\rangle \\
&= -\left\langle fT, \varphi' \right\rangle \\
&= \left\langle \frac{d}{dx}(fT), \varphi \right\rangle.
\end{aligned}$$

4) *For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have*

$$\begin{aligned}
\left\langle x \operatorname{vp}\frac{1}{x}, \varphi \right\rangle &= \left\langle \operatorname{vp}\frac{1}{x}, x\varphi \right\rangle \\
&= \lim_{\varepsilon \to 0} \int\limits_{|x|>\varepsilon} \varphi(x)\, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \left\{ \int\limits_{-\infty}^{-\varepsilon} \varphi(x)\, \mathrm{d}x + \int\limits_{\varepsilon}^{+\infty} \varphi(x)\, \mathrm{d}x \right\} \\
&= \int\limits_{-\infty}^{+\infty} \varphi(x)\, \mathrm{d}x \\
&= \langle 1, \varphi \rangle
\end{aligned}$$

5) *According to 2), $x\delta = 0$. Conversely, let T be such that $xT = 0$, then for all $\varphi \in \mathcal{D}(\mathbb{R})$*

$$\langle xT, \varphi \rangle = 0,$$

thus

$$\langle T, x\varphi\rangle = 0.$$

But ψ *is of the form* $x\varphi$ *if and only if* $\langle \delta, \psi\rangle = 0$.

6) *We have*

$$xT = 1,$$

As $x\,\mathrm{vp}\,\frac{1}{x} = 1$, *then*

$$x\left(T - \mathrm{vp}\,\frac{1}{x}\right) = 0,$$

so,

$$T = \mathrm{vp}\,\frac{1}{x} + c\delta.$$

Correction of Exercise 25

1) *We have*

$$T' - cT = \delta,$$

by multiplying both sides by e^{-cx}, *we find*

$$e^{-cx}\left(T' - cT\right) = e^{-cx}\delta$$

as $e^{-cx}\delta = \delta$, *then*

$$e^{-cx}\left(T' - cT\right) = \delta,$$

so,

$$\left(e^{-cx}T\right)' = \delta,$$

but $H' = \delta$, *so,*

$$e^{-cx}T = H + \mathrm{c},$$

we deduce that

$$T = e^{cx}(H + c).$$

2) *By multiplying both sides of the equation* $T' + xT = \delta$ *by* $e^{-\frac{x^2}{2}}$, *we obtain*

$$(T' + xT)\, e^{\frac{x^2}{2}} = \delta e^{\frac{x^2}{2}}.$$

As $\delta e^{\frac{x^2}{2}} = \delta$, *then*

$$\left(T e^{\frac{x^2}{2}}\right)' = \delta,$$

we deduce that

$$T e^{\frac{x^2}{2}} = H(x) + \mathrm{c},$$

because $H' = \delta$, *thus*

$$T = (H(x) + \mathrm{c}) e^{-\frac{x^2}{2}}.$$

3) *We multiply both sides of the equation by* e^x

$$(T' + xT)\, e^x = H(x) e^x,$$

then

$$(u e^x)' = H(x) e^x,$$

consequently

$$T e^x = \int\limits_{-\infty}^{x} H(t) e^t \,\mathrm{d}t + \mathrm{c}.$$

As

$$\int\limits_{-\infty}^{x} H(t) e^t \,\mathrm{d}t = \begin{cases} e^x - 1 & \textit{if} \quad x \geqslant 0, \\ 0 & \textit{if} \quad x < 0, \end{cases}$$

then

$$T = \left(H(x)\left(e^x - 1\right) + \mathrm{c}\right) e^{-x}.$$

4) *We multiply both sides of the equation by* $e^{e^{-x}}$, *thus*

$$\left(T' + e^{-x} T\right) e^{-e^{-x}} = e^{-e^{-x}} \delta',$$

so

$$\left(T e^{-e^{-x}}\right)' = e^{-e^{-x}} \delta'.$$

Moreover

$$f(x)\delta' = -f'(0)\delta + f(0)\delta'.$$

For $f(x) = e^{-e^{-x}}$, *we find*

$$e^{-e^{-x}}\delta' = -e^{-1}\delta + e^{-1}\delta',$$

hence

$$Te^{-e^{-x}} = -e^{-1}H(x) + e^{-1}\delta + c,$$

then

$$T = \left(-e^{-1}H(x) + e^{-1}\delta + c\right)e^{e^{-x}}.$$

5) *Let* F *be an antiderivative of* f, *we have*

$$T' + f(x)T = u,$$

by multiplying both sides by $e^{F(x)}$, *we find*

$$\left(T' + f(x)T\right)e^{F(x)} = ue^{F(x)},$$

then

$$\left(Te^{F(x)}\right)' = ue^{F(x)},$$
$$T = e^{-F(x)}\int u(x)e^{F(x)}\,\mathrm{d}x.$$

6) *Let's look for a particular solution of* $T'' - c^2T = \delta$ *in the form of* $T(x) = H(x)f(x)$, *where* f *is of class* C^2. *As* $H' = \delta$, $H'' = \delta'$, *then*

$$\begin{aligned} T'' - c^2T &= H\left(f' - c^2 f\right) + f(0)\delta + f(0)\delta' \\ &= \delta, \end{aligned}$$

by identification $f' - c^2 f = 0, f(0) = 1$ *and* $f(0) = 0, f'$, *thus*

$$f(x) = \frac{\sinh(cx)}{c}.$$

We just need to add the general solution of the associated homogeneous equation

$$T(x) = C_1\cosh(cx) + C_2\sinh(cx) + H(x)\frac{\sinh(cx)}{c}.$$

Correction of Exercise 26

Let $T = \delta'$*, then for all* φ *in* $\mathcal{D}(\mathbb{R})$ *such that* $\varphi(0) = 0$ *and* $\varphi'(0) \neq 0$*, we have*

$$\langle T, \varphi \rangle = -\varphi'(0) \neq 0.$$

More generally, $\langle T, \psi \rangle = 0$ *as soon as* ψ *is zero in the vicinity of the support of* T*, meaning that the supports of* T *and* ψ *are disjoint. We have* $\langle T, \psi_1 \rangle = \langle T, \psi_2 \rangle$ *as long as* ψ_1 *and* ψ_2 *are equal in the vicinity of the support of* T*.*

Correction of Exercise 27

1) $\operatorname{supp} T_1 = \{0\}$.

2) *Let* $F = [-1, 1]$*, and let's show that* $\operatorname{supp} T_2 = [-1, 1]$*. This amounts to demonstrating that* $w =]-\infty, -1[\cup]1, +\infty[$ *is the largest open set where* T_2 *is zero.*

   - $\star$ *w is open since it's a union of two open sets.*
   - $\star$ T_2 *is zero on* w*, indeed, for all* $\varphi \in \mathcal{D}(w)$*, we have* $\operatorname{supp} \varphi \subset w$*, and* $\langle T_2, \varphi \rangle = 0$.
   - $\star$ *w is the largest open set where* T_3 *is zero. We have*

$$\int_{-1}^{1} x\varphi'(x)\mathrm{d}x = \varphi(1) - \varphi(-1) - \int_{-1}^{1} \varphi(x)\mathrm{d}x.$$

So,

$$\langle T_2, \varphi \rangle = \langle \delta_1, \varphi \rangle - \langle \delta_{-1}, \varphi \rangle - \langle T_f, \varphi \rangle,$$

where

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [-1, 1], \\ 0 & \text{if} \quad x \notin [-1, 1], \end{cases}$$

Let U *be an open set in* $\mathbb{R}$ *larger than* w*. Then there exists* $\varphi \in \mathcal{D}(U \cap] -$

$1, 1[)$ *such that* $\varphi(x) > 0$, *and* $\varphi(1) = \varphi(-1) = 0$. *Hence,*

$$\begin{aligned}\langle T_2, \varphi\rangle &= \langle \delta_1, \varphi\rangle - \langle \delta_{-1}, \varphi\rangle - \int_{-1}^{1} \varphi(x)\mathrm{d}x \\ &= -\int_{-1}^{1} \varphi(x)\mathrm{d}x < 0,\end{aligned}$$

Thus, w *is the largest open set in which* T_2 *is zero.*

3) *Let* $F = \mathbb{R}$, *and let's show that* $\operatorname{supp} T_3 = \mathbb{R}$. *This amounts to demonstrating that* $w = \emptyset$ *is the largest open set where* T_3 *is zero.*

- ⋆ w *is open.*
- ⋆ T_3 *is zero in* w.
- ⋆ *Let* U *be an open non-empty set in* $\mathbb{R}$ *larger than* w. *Then there exist two real numbers* a *and* b *such that* $[a, b] \subset U$, *and* $\cos x > 0$ *or* $\cos x < 0$ *on* $[a, b]$. *There also exists a function* $\varphi \in \mathcal{D}(]a, b[)$ *such that* $\varphi(x) > 0$. *Therefore,*

$$\langle T_3, \varphi\rangle \neq 0,$$

which implies that T_3 *does not vanish on any non-empty open set.*

4) *We have*

$$\begin{aligned}\langle T_4, \varphi\rangle &= -\int_{-1}^{0} \varphi'(x)\mathrm{d}x + \int_{0}^{1} \varphi'(x)\mathrm{d}x \\ &= \varphi(-1) - 2\varphi(0) - \varphi(-1) \\ &= \langle \delta_{-1} - 2\delta + \delta_1, \varphi\rangle.\end{aligned}$$

Let $F = \{-1, 0, 1\}$, *and* $w = \mathbb{R} \setminus \{-1, 0, 1\}$. *It can be shown that* w *is the largest open set in which* T_4 *is zero. Therefore,* $\operatorname{supp} T_4 = \{-1, 0, 1\}$.

Correction of Exercise 28

1) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$*. We have*

$$\langle T_1, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi\left(x^2\right)\,\mathrm{d}x$$
$$= \int\limits_{0}^{\sqrt{a}} \varphi\left(x^2\right)\,\mathrm{d}x.$$

Using the change of variable $y = x^2$*, we obtain*

$$T(\varphi) = \int_0^a \varphi(y)\frac{1}{\sqrt{y}}\mathrm{d}y$$

T *is a regular distribution associated with the function* $f : x \mapsto \frac{1}{\sqrt{x}}H(x)$*. It is of order* 0 *and its support is* $\operatorname{supp} T_1 = \mathbb{R}^+$*.*

2) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*, there exists* $a > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$*. Integration by parts gives*

$$\langle T_2, \varphi\rangle = \int\limits_{-\infty}^{+\infty} \varphi'(x)\cos x\,\mathrm{d}x$$
$$= \int_{-a}^{a} \varphi'(x)e^{x^2}dx$$
$$= \left[\varphi(x)e^{x^2}\right]_{-a}^{a} - \int_{-a}^{a} 2x\varphi(x)e^{x^2}dx$$

T *is a regular distribution associated with the function* $f : x \mapsto -2xe^{x^2}$*.*

3) *Let* $\varphi \in \mathcal{D}(\mathbb{R})$*. Integration by parts gives*

$$\langle T_3, \varphi\rangle = \int\limits_{0}^{\pi} \varphi\left(x^2\right)\,\mathrm{d}x$$
$$= [\varphi(x)\cos x]_0^\pi + \int_0^\pi \varphi(x)\sin xx\,\mathrm{d}x$$
$$= -\varphi(\pi) - \varphi(0) + \int_0^\pi \varphi(x)\sin x\,\mathrm{d}x.$$

$T_3 = -\delta_\pi - \delta_0 + T_f$ *where* $f : x \mapsto \sin x\Xi_{[0,\pi]}(x)$*, this distribution is not regular. It can be easily shown that its support is* $[0, \pi]$*.*

4) *Let $a>0$, for all $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi \subset [-a,a]$, we have*

$$|\langle T_4, \varphi\rangle| \leqslant \int_{-\infty}^{+\infty} |\varphi'(x)|\,|\ln x|\,\mathrm{d}x$$
$$\leqslant \sup|\varphi'| \int_0^A |\ln x|\,\mathrm{d}x$$

T is a distribution of order less than or equal to 1. Let's show that the order of T is not 0. Let the sequence $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n(x) = \begin{cases} -1 & \text{if } \; x \in \left[\frac{1}{n}, 1\right] \\ 0 & \text{if } \; x \notin \left[\frac{1}{2n}, 2\right]. \end{cases}$$

$\varphi_n \in \mathcal{D}([0,2])$ and $-1 \leqslant \varphi_n \leqslant 0$.

Assume that T is of order 0. Then, there exists $C>0$ such that

$$|\langle T_4, \varphi_n\rangle| \leqslant C \sup|\varphi_n| \leqslant C.$$

Furthermore, φ_n and the function $\ln$ are of class $\mathcal{C}^1$ on $[\frac{1}{2n}, 2]$, and

$$\langle T_4, \varphi_n\rangle = \int_{\frac{1}{2n}}^{2} \varphi_n'(x)\ln x dx$$
$$= -\int_{\frac{1}{2n}}^{2} \frac{\varphi_n(x)}{x} dx$$
$$\geqslant \int_{\frac{1}{n}}^{1} \frac{1}{x} dx$$
$$\geqslant \ln n.$$

So, $\ln n \leqslant C$ for all n, which leads to a contradiction.

As for the support of T_4, it is clear that $\operatorname{supp} T \subset \mathbb{R}^+$. Furthermore, if $x_0>0$, there exists $\delta>0$ such that $]x_0-\delta, x_0+\delta[\subset]0,+\infty[$. So, if φ belongs to $\mathcal{D}(]x_0-\delta, x_0+\delta[)$, we can calculate $\langle T, \varphi\rangle$ through integration by parts, for $x_0 \in \operatorname{supp} T$. Therefore $]0,+\infty[\subset \operatorname{supp} T$, and as $\operatorname{supp} T$ is closed, we deduce that $\operatorname{supp} T = \mathbb{R}^+$.

Correction of Exercise 29

1) *Let K be a compact subset of $\mathbb{R}^2$ and $\varphi \in \mathcal{D}_K(\mathbb{R}^2)$. We have*

$$\begin{aligned}\left|\int_0^1 \varphi(x,x+1)\,\mathrm{d}x\right| &= \int_0^1 |\varphi(x,x+1)|\,\mathrm{d}x\\ &\leqslant \sup_{(x,y)\in K} |\varphi(x,y)|\\ &\leqslant P_{K,0}(\varphi).\end{aligned}$$

T is a distribution of order 0.

2) *Let $\Delta \subset \mathbb{R}^2$ such that*

$$\{(x,x+1);\ 0 \leqslant x \leqslant 1\}.$$

Let $w = \mathbb{R}^2 \setminus \Delta$. w is an open set, and for all $\mathcal{D}(w)$, we have

$$\langle T,\varphi\rangle = \int_0^1 \varphi(x,x+1)\,\mathrm{d}x = 0.$$

Suppose that there exists an open set U larger than w such that T is zero in U, then there exists $r > 0$, $B(a,r) \cap U$, with $B(a,r) \subset U$, and a test function $\varphi > 0$ and $\varphi = 1$ in the neighborhood of a. Then there exists $\varepsilon > 0$ such that

$$\varphi(x,x+1) = 1, \quad \forall x \in [a-\varepsilon, a+\varepsilon]$$

Therefore,

$$\langle T,\varphi\rangle = \int_{a-\varepsilon}^{a+\varepsilon} \varphi(x,x+1)\,\mathrm{d}x = \int_{a-\varepsilon}^{a+\varepsilon} 1\,\mathrm{d}x = 2\varepsilon > 0.$$

which is absurd, so $\operatorname{supp} T = \Delta$.

Correction of Exercise 30

1) *Let K be a compact subset of $\mathbb{R}^2$ and $\varphi \in \mathcal{D}_K(\mathbb{R}^2)$. Let $K_1 = \{x \in \mathbb{R},\ (x,-x) \in K\}$, which is compact since K is. We have*

$$|\langle T,\varphi\rangle| \leqslant \left(\int_{K_1} dx\right) \sup_{(x,y)\in K} |\varphi(x,y)|.$$

T is a distribution of order 0.

2) *Let* $\Delta \subset \mathbb{R}^2$ *such that*

$$\Delta = \{(x, -x),\ x \in \mathbb{R}\}.$$

Let $w = \mathbb{R}^2 \setminus \Delta$. w *is an open set, and for all* φ *in* $\mathcal{D}(w)$, *we have*

$$\langle T, \varphi \rangle = 0.$$

Suppose that there exists an open set U *larger than* w *such that* T *is zero in* U, *then there exists* $r > 0$, $B(a,r) \cap U$, *with* $B(a,r) \subset U$, *and a test function* $\varphi > 0$ *and* $\varphi = 1$ *in the neighborhood of* a. *Then there exists* $\varepsilon > 0$ *such that*

$$\varphi(x, -x) = 1, \quad \forall x \in [a - \varepsilon, a + \varepsilon]$$

Therefore,

$$\langle T, \varphi \rangle = \int_{a-\varepsilon}^{a+\varepsilon} \varphi(x, -x)\, \mathrm{d}x = \int_{a-\varepsilon}^{a+\varepsilon} 1\, \mathrm{d}x = 2\varepsilon > 0.$$

which is absurd, thus $\operatorname{supp} T = \Delta$.

3) *Let* $\psi(x) = \varphi(x, -x)$. *Then*

$$\psi'(x) = \frac{\partial \varphi}{\partial x}(x, -x) - \frac{\partial \varphi}{\partial y}(x, -x),$$

then

$$\left\langle \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}, \varphi \right\rangle = -\int_{\mathbb{R}} \left(\frac{\partial \varphi}{\partial x}(x, -x) - \frac{\partial \varphi}{\partial y}(x, -x) \right) dx = [\psi(x)]_{-\infty}^{+\infty} = 0.$$

BIBLIOGRAPHY

[1] J. M. Bony, Cours d'analyse. Théorie des distributions et analyse de Fourier, Ellipses, (2001).

[2] A. Bouziad, J. Calbrix, Théorie de la mesure et de l'intégration, Publications de l'Université de Rouen, (1993).

[3] H. Brezis, Analyse fonctionnelle, Théorie et applications, Masson, Paris, (1983).

[4] G. Carlier, Notes de cours : Analyse fonctionnelle, https://www.ceremade.dauphine.fr/carlier/poly2010.pdf

[5] R. Jonathan, Les espaces de Sobolev, École polytéchnique fédérale de lausanne, SMA, (2009).

[6] G. Friedlander, M. Joshi, Introduction to the theory of distributions, Cambridge University. Press. (1998).

[7] L. Hörmander, Distribution theory and Fourier analysis, Springer, (2000).

[8] J.P. Marco et autres, Mathematiques L3, Analyse, Pearson Education France.

[9] B. Simon et M. Reed, Methods of modern mathematical physics. II. Fourier analysis, selfadjointness, Academic Press, New York-London, (1975).

[10] C. Zuily, Distributions et équations aux dérivées partielles, exercices corrigés, Hermann, ***157*** *(1986).*

[11] C. Zuily, Eléments de distributions et équations aux dérivées partielles, Dunod, (2002).

[12] T. Gallay, Théorie de la mesure et de l'intégration, Université Joseph Fourier, Grenoble, (2009).